INTEGRABLE SYSTEMS FROM MEMBRANES ON $AdS_4 \times S^7$

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1 INTRODUCTION

The 2-branes (membranes) and 5-branes are the fundamental dynamical objects in the eleven dimensional M-theory, which is the strong coupling limit of the five superstring theories in ten dimensions, and which low energy field theory limit is the eleven dimensional supergravity.

It is known that large class of classical string solutions in the type IIB $AdS_5 \times S^5$ background is related to the Neumann and Neumann-Rosochatius integrable systems, including recently discovered spiky strings and giant magnons. It is also interesting if these integrable systems can be associated with some membrane configurations in M-theory. I will talk about how this can be achieved by considering several types of membrane embedding in $AdS_4 \times S^7$ solution of M-theory, with the desired properties.

On the other hand, I could show you the existence of membrane configurations in $AdS_4 \times S^7$, which correspond to the continuous limit of the $SU(2)$ integrable spin chain, considered as a limit of the $SU(3)$ spin chain, both arising in $\mathcal{N} = 4$ SYM in four dimensions, dual to strings in $AdS_5 \times S^5$.

2 MEMBRANES ON $AdS_4 \times S^7$

The membrane action and constraints in diagonal gauge

$$
\begin{aligned}
&S_M = \int d^3\xi \mathcal{L}_M = \\
&\int d^3\xi \left\{ \frac{1}{4\lambda^0}[G_{00} - \left(2\lambda^0 T_2\right)^2 \det G_{ij}] + T_2 C_{012} \right\}, \\
&G_{00} + \left(2\lambda^0 T_2\right)^2 \det G_{ij} = 0, \\
&G_{0i} = 0.
\end{aligned}
$$

They *coincide* with the frequently used gauge fixed Polyakov type action and constraints after the identification $2\lambda^0 T_2 = L = const.$ The fields induced on the membrane worldvolume G_{mn} and C_{012} are given by

$$
\begin{aligned}
&G_{mn} = g_{MN}\partial_m X^M \partial_n X^N, \\
&C_{012} = c_{MNP}\partial_0 X^M \partial_1 X^N \partial_2 X^P, \\
&\partial_m = \partial/\partial\xi^m, \quad m = (0, i) = (0, 1, 2), \\
&(\xi^0, \xi^1, \xi^2) = (\tau, \sigma_1, \sigma_2), \quad M = (0, 1, \ldots, 10),
\end{aligned}
$$

where g_{MN} and c_{MNP} are the components of the target space metric and 3-form gauge field respectively.

Searching for membrane configurations in $AdS_4 \times S^7$, which correspond to the Neumann or Neumann-Rosochatius integrable systems, we should first eliminate the membrane interaction with the background 3-form field on AdS_4, to ensure more close analogy with the strings on $AdS_5 \times S^5$. To make our choice, let us write down the background. It can be parameterized as follows

$$
\begin{aligned}
&ds^2 = (2l_p\mathcal{R})^2 \left[-\cosh^2\rho dt^2 + d\rho^2 \right.\\
&\left. + \sinh^2\rho \left(d\alpha^2 + \sin^2\alpha d\beta^2\right) + 4d\Omega_7^2\right], \\
&c_{(3)} = (2l_p\mathcal{R})^3 \sinh^3\rho \sin\alpha dt \wedge d\alpha \wedge d\beta.
\end{aligned}
$$

Since we want the membrane to have nonzero conserved energy and spin on AdS, the possible choice, for which the interaction with the $c_{(3)}$ field disappears, is to fix the angle α:

$$\alpha = \alpha_0 = const.$$

The metric of the corresponding subspace of AdS_4 is

$$ds^2_{sub} = (2l_p\mathcal{R})^2\left[-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d(\beta\sin\alpha_0)^2\right].$$

The appropriate membrane embedding into ds^2_{sub} and S^7 is

$$\begin{aligned}
Z_\mu &= 2l_p\mathcal{R}\mathrm{r}_\mu(\xi^m)e^{i\phi_\mu(\xi^m)},\\
&\quad \mu = (0,1),\quad \phi_\mu = (\phi_0,\phi_1) = (t,\beta\sin\alpha_0),\\
&\quad \eta^{\mu\nu}\mathrm{r}_\mu\mathrm{r}_\nu + 1 = 0,\quad \eta^{\mu\nu} = (-1,1),\\
W_a &= 4l_p\mathcal{R}r_a(\xi^m)e^{i\varphi_a(\xi^m)},\\
&\quad a = (1,2,3,4),\quad \delta_{ab}r_ar_b - 1 = 0.
\end{aligned}$$

For this embedding, the induced metric is given by

$$\begin{aligned}
G_{mn} &= \eta^{\mu\nu}\partial_{(m}Z_\mu\partial_{n)}\bar{Z}_\nu + \delta_{ab}\partial_{(m}W_a\partial_{n)}\bar{W}_b =\\
&(2l_p\mathcal{R})^2\left[\sum_{\mu,\nu=0}^{1}\eta^{\mu\nu}\left(\partial_m\mathrm{r}_\mu\partial_n\mathrm{r}_\nu + \mathrm{r}_\mu^2\partial_m\phi_\mu\partial_n\phi_\nu\right)\right.\\
&\left. + 4\sum_{a=1}^{4}\left(\partial_m r_a\partial_n r_a + r_a^2\partial_m\varphi_a\partial_n\varphi_a\right)\right].
\end{aligned}$$

Correspondingly, the membrane Lagrangian becomes

$$\mathcal{L} = \mathcal{L}_M + \Lambda_A(\eta^{\mu\nu}\mathrm{r}_\mu\mathrm{r}_\nu + 1) + \Lambda_S(\delta_{ab}r_ar_b - 1).$$

2.1 MEMBRANES AND THE NEUMANN SYSTEM

First embedding

$$Z_0 = 2l_p\mathcal{R}e^{i\kappa\tau}, \quad Z_1 = 0, \quad W_a = 4l_p\mathcal{R}r_a(\tau)e^{i\omega_{ai}\sigma_i}.$$

This implies

$$\mathrm{r}_0 = 1, \quad \mathrm{r}_1 = 0, \quad \phi_0 = \kappa\tau, \quad \varphi_a = \omega_{ai}\sigma_i.$$

Then the membrane Lagrangian takes the form (over-dot is used for $d/d\tau$)

$$\begin{aligned}\mathcal{L} &= \frac{(4l_p\mathcal{R})^2}{4\lambda^0}\left[\sum_{a=1}^{4}\dot{r}_a^2 - (\kappa/2)^2\right.\\ &- \left.\left(8\lambda^0 T_2 l_p\mathcal{R}\right)^2\sum_{a<b=1}^{4}(\omega_{a1}\omega_{b2} - \omega_{a2}\omega_{b1})^2 r_a^2 r_b^2\right]\\ &+ \Lambda_S\left(\sum_{a=1}^{4} r_a^2 - 1\right).\end{aligned}$$

As far as we are interested in obtaining membrane configurations with quadratic effective potential, our proposal is to make the following choice ($a, b, c \neq 0$ are constants)

$$\begin{aligned}&\omega_{12} = \omega_{22} = \omega_{31} = \omega_{41} = 0, \quad \omega_{32} = \pm\omega_{42} = \omega,\\ &r_3(\tau) = a\sin(b\tau + c), \quad r_4(\tau) = a\cos(b\tau + c), \quad a < 1.\end{aligned}$$

This reduces $\mathcal{L}$ to (after neglecting the constat terms)

$$\begin{aligned}L &= \frac{(4l_p\mathcal{R})^2}{4\lambda^0}\sum_{a=1}^{2}\left[\dot{r}_a^2 - \left(8\lambda^0 T_2 l_p\mathcal{R}a\omega\right)^2\omega_{a1}^2 r_a^2\right]\\ &+ \Lambda_S\left[\sum_{a=1}^{2} r_a^2 - (1 - a^2)\right].\end{aligned}$$

The Lagrangian L describes two-dimensional harmonic oscillator, constrained to remain on a circle of radius $\sqrt{1 - a^2}$. This is particular case of the Neumann integrable system.

The first constraint gives the Hamiltonian corresponding to L

$$H \sim \sum_{a=1}^{2} \left[\dot{r}_a^2 + \left(8\lambda^0 T_2 l_p \mathcal{R} a\omega\right)^2 \omega_{a1}^2 r_a^2\right] = (\kappa/2)^2 - (ab)^2,$$

while the remaining two constraints are satisfied identically.

Second embedding

$$Z_0 = 2l_p \mathcal{R} e^{i\kappa\tau}, \quad Z_1 = 0, \quad W_a = 4l_p \mathcal{R} r_a(\sigma_i) e^{i\omega_a \tau}.$$

Now the membrane Lagrangian $\mathcal{L}$ is given by

$$\begin{aligned}\mathcal{L} &= -\frac{(4l_p\mathcal{R})^2}{4\lambda^0}\left[\left(8\lambda^0 T_2 l_p \mathcal{R}\right)^2 \sum_{a<b=1}^{4} (\partial_1 r_a \partial_2 r_b - \partial_2 r_a \partial_1 r_b)^2\right. \\ &- \left.\sum_{a=1}^{4} \omega_a^2 r_a^2 + (\kappa/2)^2\right] + \Lambda_S \left(\sum_{a=1}^{4} r_a^2 - 1\right).\end{aligned}$$

Here we have quadratic potential, but in the general case, the kinetic term is not of the type we are searching for. To fix the problem, we set

$$\begin{aligned}&r_1 = r_1(\sigma_1), \quad r_2 = r_2(\sigma_1), \quad \omega_3 = \pm\omega_4 = \omega, \\ &r_3(\sigma_2) = a\sin(b\sigma_2 + c), \quad r_4(\sigma_2) = a\cos(b\sigma_2 + c), \quad a < 1.\end{aligned}$$

This leads to the Lagrangian (prime is used for $d/d\sigma_1$)

$$\begin{aligned}L &= \frac{(4l_p\mathcal{R})^2}{4\lambda^0} \sum_{a=1}^{2} \left[\left(8\lambda^0 T_2 l_p \mathcal{R} ab\right)^2 r_a'^2 - \omega_a^2 r_a^2\right] \\ &+ \Lambda_S \left[\sum_{a=1}^{2} r_a^2 - (1 - a^2)\right],\end{aligned}$$

which is already of the Neumann type. The corresponding Hamiltonian is given by the first constraint

$$H \sim \sum_{a=1}^{2} \left[\left(8\lambda^0 T_2 l_p \mathcal{R} ab\right)^2 r_a'^2 + \omega_a^2 r_a^2\right] = (\kappa/2)^2 - (a\omega)^2.$$

The other two constraints are satisfied identically.

2.2 MEMBRANES AND THE NEUMANN-ROSOCHATIUS SYSTEM

Third embedding

$$Z_0 = 2l_p\mathcal{R}e^{i\kappa\tau}, \quad Z_1 = 0, \quad W_a = 4l_p\mathcal{R}r_a(\tau)e^{i[\omega_{ai}\sigma_i+\alpha_a(\tau)]}.$$

It leads to the following membrane Lagrangian

$$\begin{aligned}\mathcal{L} &= \frac{(4l_p\mathcal{R})^2}{4\lambda^0}\left[\sum_{a=1}^{4}\left(\dot{r}_a^2 + r_a^2\dot{\alpha}_a^2\right)\right.\\ &- \left.\left(8\lambda^0 T_2 l_p\mathcal{R}\right)^2 \sum_{a<b=1}^{4}(\omega_{a1}\omega_{b2} - \omega_{a2}\omega_{b1})^2 r_a^2 r_b^2 - (\kappa/2)^2\right]\\ &+ \Lambda_S\left(\sum_{a=1}^{4} r_a^2 - 1\right).\end{aligned}$$

The equations of motion for the variables $\alpha_a(\tau)$ can be easily integrated once and the result is

$$\dot{\alpha}_a(\tau) = \frac{C_a}{r_a^2(\tau)},$$

where C_a are arbitrary integration constants. Substituting this back into $\mathcal{L}$, one receives an effective Lagrangian for the four real coordinates $r_a(\tau)$

$$\begin{aligned}\mathcal{L} &= \frac{(4l_p\mathcal{R})^2}{4\lambda^0}\left[\sum_{a=1}^{4}\left(\dot{r}_a^2 - \frac{C_a^2}{r_a^2}\right)\right.\\ &- \left.\left(8\lambda^0 T_2 l_p\mathcal{R}\right)^2 \sum_{a<b=1}^{4}(\omega_{a1}\omega_{b2} - \omega_{a2}\omega_{b1})^2 r_a^2 r_b^2 - (\kappa/2)^2\right]\\ &+ \Lambda_S\left(\sum_{a=1}^{4} r_a^2 - 1\right).\end{aligned}$$

To get potential terms $\sim r_a^2$ instead of $\sim r_a^2 r_b^2$, we use once again the choice made for the first embedding. In addition, we put

$C_3 = C_4 = 0$. All this reduces the membrane Lagrangian to (after neglecting the constant terms)

$$L = \frac{(4l_p\mathcal{R})^2}{4\lambda^0}\sum_{a=1}^{2}\left[\dot{r}_a^2 - \left(8\lambda^0 T_2 l_p \mathcal{R} a\omega\right)^2 \omega_{a1}^2 r_a^2 - \frac{C_a^2}{r_a^2}\right] + \Lambda_S\left[\sum_{a=1}^{2} r_a^2 - (1-a^2)\right],$$

which describes Neumann-Rosochatius integrable system.

The constraints are

$$H \sim \sum_{a=1}^{2}\left[\dot{r}_a^2 + \left(8\lambda^0 T_2 l_p \mathcal{R} a\omega\right)^2 \omega_{a1}^2 r_a^2 + \frac{C_a^2}{r_a^2}\right] = (\kappa/2)^2 - (ab)^2,$$

$$\sum_{a=1}^{2}\omega_{a1}C_a = 0, \quad G_{02} \equiv 0.$$

Forth embedding

$$Z_0 = 2l_p\mathcal{R}e^{i\kappa\tau}, \quad Z_1 = 0, \quad W_a = 4l_p\mathcal{R}r_a(\sigma_i)e^{i[\omega_a\tau + \alpha_a(\sigma_i)]},$$

for which the membrane Lagrangian reduces to

$$\begin{aligned}\mathcal{L} = & -\frac{(4l_p\mathcal{R})^2}{4\lambda^0}\left\{\left(8\lambda^0 T_2 l_p \mathcal{R}\right)^2 \sum_{a<b=1}^{4}\left[(\partial_1 r_a \partial_2 r_b - \partial_2 r_a \partial_1 r_b)^2\right.\right.\\ & + (\partial_1 r_a \partial_2 \alpha_b - \partial_2 r_a \partial_1 \alpha_b)^2 r_b^2 + (\partial_1 \alpha_a \partial_2 r_b - \partial_2 \alpha_a \partial_1 r_b)^2 r_a^2\\ & \left. + (\partial_1 \alpha_a \partial_2 \alpha_b - \partial_2 \alpha_a \partial_1 \alpha_b)^2 r_a^2 r_b^2\right]\\ & + \sum_{a=1}^{4}\left[\left(8\lambda^0 T_2 l_p \mathcal{R}\right)^2 (\partial_1 r_a \partial_2 \alpha_a - \partial_2 r_a \partial_1 \alpha_a)^2 - \omega_a^2\right] r_a^2\\ & \left. + (\kappa/2)^2\right\} + \Lambda_S\left(\sum_{a=1}^{4} r_a^2 - 1\right).\end{aligned}$$

If we restrict ourselves to the case as for the second embedding and

$$\alpha_1 = \alpha_1(\sigma_1), \quad \alpha_2 = \alpha_2(\sigma_1), \quad \alpha_3, \alpha_4 = constants,$$

we obtain

$$\mathcal{L} = -\frac{(4l_p\mathcal{R})^2}{4\lambda^0}\left[\left(8\lambda^0 T_2 l_p \mathcal{R} ab\right)^2 \sum_{a=1}^{2}\left(r_a'^2 + r_a^2\alpha_a'^2\right)\right.$$

$$- \sum_{a=1}^{2} \omega_a^2 r_a^2 + (\kappa/2)^2 - (a\omega)^2 \Big].$$

After integrating the equations of motion for α_a once and replacing the solution into the Lagrangian, one arrives at

$$L = \frac{(4l_p\mathcal{R})^2}{4\lambda^0} \sum_{a=1}^{2} \Big[\left(8\lambda^0 T_2 l_p \mathcal{R} ab\right)^2 r_a'^2 - \omega_a^2 r_a^2 - \left(8\lambda^0 T_2 l_p \mathcal{R} ab\right)^2 \frac{C_a^2}{r_a^2} \Big] + \Lambda_S \left[\sum_{a=1}^{2} r_a^2 - (1 - a^2) \right].$$

The above Lagrangian represents particular case of the Neumann-Rosochatius integrable system.

The constraints for the case under consideration are given by

$$H \sim \sum_{a=1}^{2} \left[\left(8\lambda^0 T_2 l_p \mathcal{R} ab\right)^2 r_a'^2 + \omega_a^2 r_a^2 + \left(8\lambda^0 T_2 l_p \mathcal{R} ab\right)^2 \frac{C_a^2}{r_a^2} \right] = (\kappa/2)^2 - (a\omega)^2,$$

$$\sum_{a=1}^{2} \omega_a C_a = 0, \quad G_{02} = 0.$$

Fifth embedding
(connected to the spiky strings and giant magnons)

$$Z_0 = 2l_p\mathcal{R}e^{i\kappa\tau}, \quad Z_1 = 0,$$
$$W_a = 4l_p\mathcal{R} r_a(\xi,\eta) e^{i[\omega_a\tau + \mu_a(\xi,\eta)]},$$
$$\xi = \alpha\sigma_1 + \beta\tau, \quad \eta = \gamma\sigma_2 + \delta\tau,$$

where α, β, γ, δ are constants. For this ansatz, the membrane Lagrangian takes the form ($\partial_\xi = \partial/\partial\xi$, $\partial_\eta = \partial/\partial\eta$)

$$\mathcal{L} = -\frac{(4l_p\mathcal{R})^2}{4\lambda^0} \Big\{ \left(8\lambda^0 T_2 l_p \mathcal{R} \alpha\gamma\right)^2 \sum_{a<b=1}^{4} \Big[(\partial_\xi r_a \partial_\eta r_b - \partial_\eta r_a \partial_\xi r_b)^2 + (\partial_\xi r_a \partial_\eta \mu_b - \partial_\eta r_a \partial_\xi \mu_b)^2 r_b^2 + (\partial_\xi \mu_a \partial_\eta r_b - \partial_\eta \mu_a \partial_\xi r_b)^2 r_a^2 + (\partial_\xi \mu_a \partial_\eta \mu_b - \partial_\eta \mu_a \partial_\xi \mu_b)^2 r_a^2 r_b^2 \Big]$$

$$+ \sum_{a=1}^{4} \left[\left(8\lambda^0 T_2 l_p \mathcal{R} \alpha\gamma\right)^2 (\partial_\xi r_a \partial_\eta \mu_a - \partial_\eta r_a \partial_\xi \mu_a)^2 \right.$$
$$\left. - (\beta\partial_\xi \mu_a + \delta\partial_\eta \mu_a + \omega_a)^2 \right] r_a^2 - \sum_{a=1}^{4} (\beta\partial_\xi r_a + \delta\partial_\eta r_a)^2$$
$$+ (\kappa/2)^2 \Big\} + \Lambda_S \left(\sum_{a=1}^{4} r_a^2 - 1 \right).$$

Now, we choose to consider the particular case

$$r_1 = r_1(\xi), \quad r_2 = r_2(\xi), \quad \omega_3 = \pm\omega_4 = \omega,$$
$$r_3 = r_3(\eta) = a\sin(b\eta + c), \quad r_4 = r_4(\eta) = a\cos(b\eta + c),$$
$$\mu_1 = \mu_1(\xi), \quad \mu_2 = \mu_2(\xi), \quad \mu_3, \mu_4 = constants,$$

and receive (prime is used for $d/d\xi$)

$$\mathcal{L} = -\frac{(4l_p\mathcal{R})^2}{4\lambda^0} \left\{ \sum_{a=1}^{2} \left[(A^2 - \beta^2) r_a'^2 \right. \right.$$
$$\left. + (A^2 - \beta^2) r_a^2 \left(\mu_a' - \frac{\beta\omega_a}{A^2 - \beta^2} \right)^2 - \frac{A^2}{A^2 - \beta^2} \omega_a^2 r_a^2 \right]$$
$$\left. + (\kappa/2)^2 - a^2(\omega^2 + b^2\delta^2) \right\} + \Lambda_S \left[\sum_{a=1}^{2} r_a^2 - (1 - a^2) \right],$$

where

$$A^2 \equiv \left(8\lambda^0 T_2 l_p \mathcal{R} ab\alpha\gamma\right)^2.$$

A single time integration of the equations of motion for μ_a following from the above Lagrangian gives

$$\mu_a' = \frac{1}{A^2 - \beta^2} \left(\frac{C_a}{r_a^2} + \beta\omega_a \right).$$

Substituting this solution back into $\mathcal{L}$, one obtains the following effective Lagrangian for the coordinates $r_a(\xi)$

$$L = \frac{(4l_p\mathcal{R})^2}{4\lambda^0} \sum_{a=1}^{2} \left[(A^2 - \beta^2) r_a'^2 - \frac{1}{A^2 - \beta^2} \frac{C_a^2}{r_a^2} - \frac{A^2}{A^2 - \beta^2} \omega_a^2 r_a^2 \right]$$
$$+ \Lambda_S \left[\sum_{a=1}^{2} r_a^2 - (1 - a^2) \right].$$

Let us write down the constraints for the present case. To achieve more close correspondence with the string on $AdS_5 \times S^5$, we want the third one to be satisfied identically. To this end, since $G_{02} \sim (ab)^2\gamma\delta$, we set $\delta = 0$, i.e. $\eta = \gamma\sigma_2$. Then, the first two constraints give

$$H \sim \sum_{a=1}^{2}\left[(A^2-\beta^2)r_a'^2 + \frac{1}{A^2-\beta^2}\frac{C_a^2}{r_a^2} + \frac{A^2}{A^2-\beta^2}\omega_a^2 r_a^2\right] = \frac{A^2+\beta^2}{A^2-\beta^2}\left[(\kappa/2)^2 - (a\omega)^2\right],$$

$$\sum_{a=1}^{2}\omega_a C_a + \beta\left[(\kappa/2)^2 - (a\omega)^2\right] = 0.$$

This Lagrangian in full analogy with the string considerations corresponds to particular case of the n-dimensional Neumann-Rosochatius integrable system.

2.3 ENERGY AND ANGULAR MOMENTA

The energy E and the angular momenta J_a can be computed by using the equalities

$$E = -\int d^2\sigma \frac{\partial \mathcal{L}}{\partial \kappa}, \quad J_a = \int d^2\sigma \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi_a)}.$$

Then, for all ansatzes we used, the energy is given by

$$E = 2^3(\pi l_p \mathcal{R})^2 \frac{\kappa}{\lambda^0}. \tag{2.1}$$

For the first embedding, $J_a = 0$ for $a = 1,2,3,4$, so the only nontrivial conserved quantity is the membrane energy.

For the second embedding, one obtains

$$J_a = 2^3(l_p\mathcal{R})^2\frac{\omega_a}{\lambda^0}\int d^2\sigma r_a^2, \quad a = 1,2,3,4.$$

We will consider the cases $a = 1,2$ and $a = 3,4$ separately. According to our ansatz, $r_{1,2} = r_{1,2}(\sigma_1)$, which leads to

$$J_a = \pi(4l_p\mathcal{R})^2\frac{\omega_a}{\lambda^0}\int d\sigma_1 r_a^2(\sigma_1), \quad a = 1,2.$$

Combining these two equalities with the expression for the energy and taking into account the constraint

$$\sum_{a=1}^{2} r_a^2 - (1-a^2) = 0,$$

one arrives at the energy-charge relation

$$\frac{E}{\kappa} = \frac{1}{4(1-a^2)}\left(\frac{J_1}{\omega_1}+\frac{J_2}{\omega_2}\right).$$

As usual, we have linear dependence $E(J_1, J_2)$ before taking the semiclassical limit.

Let us turn to the case $a = 3,4$. Now we have

$$J_3 = \pi(4l_p\mathcal{R})^2\frac{\omega a^2}{\lambda^0}\int_0^{2\pi} d\sigma_2 \sin^2(b\sigma_2 + c),$$

$$J_4 = \pm\pi(4l_p\mathcal{R})^2\frac{\omega a^2}{\lambda^0}\int_0^{2\pi} d\sigma_2 \cos^2(b\sigma_2 + c).$$

By using the periodicity conditions

$$r_a(\sigma_i) = r_a(\sigma_i + 2\pi),$$

which imply $b = \pm 1, \pm 2, \ldots$, one obtains

$$J_3 = \pm J_4 = (4\pi l_p\mathcal{R})^2\frac{\omega a^2}{\lambda^0}.$$

In order to reproduce the string case, we can set $\omega = 0$, and thus $J_3 = J_4 = 0$.

For the third embedding, the angular momenta are given by

$$J_a = 2^5(\pi l_p\mathcal{R})^2\frac{C_a}{\lambda^0}, \quad a = 1,2; \quad J_3 = J_4 = 0.$$

This leads to the energy-charge relation

$$\frac{E}{\kappa}=\frac{1}{8}\left(\frac{J_1}{C_1}+\frac{J_2}{C_2}\right).$$

For the forth membrane embedding, the expressions for the conserved charges and the relation between them are the same as for the second membrane embedding.

Finally, for the fifth membrane embedding, $J_3 = J_4 = 0$ for $\omega = 0$. The other two angular momenta are

$$J_a=\frac{\pi(4l_p\mathcal{R})^2}{\lambda^0\alpha(A^2-\beta^2)}\int d\xi\left(\beta C_a+A^2\omega_a r_a^2\right),\quad a=1,2.$$

Rewriting the energy as

$$E=\frac{4\pi(l_p\mathcal{R})^2\kappa}{\lambda^0\alpha}\int d\xi,$$

we obtain the energy-charge relation

$$\frac{4}{A^2-\beta^2}\left[A^2(1-a^2)+\beta\sum_{a=1}^{2}\frac{C_a}{\omega_a}\right]\frac{E}{\kappa}=\sum_{a=1}^{2}\frac{J_a}{\omega_a},$$

in full analogy with the string case. Namely, for strings on $AdS_5\times S^5$, the result in conformal gauge is

$$\frac{1}{\alpha^2-\beta^2}\left(\alpha^2+\beta\sum_a\frac{C_a}{\omega_a}\right)\frac{E}{\kappa}=\sum_a\frac{J_a}{\omega_a}.$$

Remark

It may seems that the membrane configurations considered here are chosen randomly. However, they correspond exactly to *all* string embeddings in the $R\times S^5$ subspace of $AdS_5\times S^5$ solution of type IIB string theory, which are known to lead the Neumann and Neumann-Rosochatius dynamical systems.

3 $SU(2)$ SPIN CHAIN FROM MEMBRANE

One of the predictions of AdS/CFT duality is that the string theory on $AdS_5 \times S^5$ should be dual to $\mathcal{N} = 4$ SYM theory in four dimensions. The spectrum of the string states and of the operators in SYM should be the same. The first checks of this conjecture *beyond* the supergravity approximation revealed that there exist string configurations, which in the semiclassical limit are related to the anomalous dimensions of certain gauge invariant operators in the planar SYM. On the field theory side, it was found that the corresponding dilatation operator is connected to the Hamiltonian of integrable Heisenberg spin chain. On the other hand, it was established that there is agreement at the level of actions between the continuous limit of the $SU(2)$ spin chain arising in $\mathcal{N} = 4$ SYM theory and a certain limit of the string action in $AdS_5 \times S^5$ background. Shortly after, it was shown that such equivalence also holds for the $SU(3)$ and $SL(2)$ cases.

<u>The question</u>

Is it possible to reproduce this type of string/spin chain correspondence from membranes on eleven dimensional curved backgrounds?

<u>The answer</u>

YES, at least for the case of M2-branes on $AdS_4 \times S^7$.

<u>**HOW?**</u>

3.1 STRINGS ON $AdS_5 \times S^5$

It is known that the ferromagnetic integrable $SU(3)$ spin chain provides the one-loop anomalous dimension of single trace operators involving the three complex scalars of $\mathcal{N} = 4$ SYM. The nonlinear sigma-model, describing the continuum limit of the $SU(3)$ spin chain, corresponds to strings moving with large angular momentum on the five-sphere in $AdS_5 \times S^5$.

In order to have more close analogy with the membrane case considered in the next section, we will reproduce the relevant string action in the framework of diagonal worldsheet gauge. In this gauge, the Polyakov action and constraints are given by

$$\begin{aligned} S_S &= \int d^2\xi \mathcal{L}_S = \int d^2\xi \frac{1}{4\lambda^0}[G_{00} - \left(2\lambda^0 T\right)^2 G_{11}], \\ G_{00} &+ \left(2\lambda^0 T\right)^2 G_{11} = 0, \\ G_{01} &= 0, \end{aligned}$$

where

$$G_{mn} = g_{MN}\partial_m X^M \partial_n X^N,$$

is the induced metric and λ^0 is Lagrange multiplier. The commonly used conformal gauge corresponds to $2\lambda^0 T = 1$.

We choose to embed the string in $AdS_5 \times S^5$ as follows

$$\begin{aligned} Z_s &= R\mathrm{r}_s(\xi^m)e^{i\phi_s(\xi^m)}, \\ & s = (0,1,2), \quad \eta^{rs}\mathrm{r}_r\mathrm{r}_s + 1 = 0, \quad \eta^{rs} = (-1,1,1), \\ W_i &= Rr_i(\xi^m)e^{i\varphi_i(\xi^m)}, \\ & i = (1,2,3), \quad \delta_{ij}r_i r_j - 1 = 0, \end{aligned}$$

where ϕ_s and φ_i are the isometric coordinates on which the metric of AdS_5 and S^5 respectively does not depend.

Here, we are interested in the following particular case of the above embedding

$$Z_0 = Re^{i\kappa\tau}, \quad Z_1 = Z_2 = 0,$$

which implies

$$\mathrm{r}_0 = 1, \quad \mathrm{r}_1 = \mathrm{r}_2 = 0; \quad \phi_0 = t = \kappa\tau, \quad \kappa = const.$$

For this ansatz, G_{mn} reduces to

$$G_{mn} = R^2 \left[\sum_{i=1}^{3} \left(\partial_m r_i \partial_n r_i + r_i^2 \partial_m \varphi_i \partial_n \varphi_i\right) - \delta_m^0 \delta_n^0 \kappa^2\right].$$

We now introduce new coordinates according to the rule

$$(\varphi_1, \varphi_2, \varphi_3) = (\kappa\tau + \alpha + \varphi, \kappa\tau + \alpha - \varphi, \kappa\tau + \alpha + \phi)$$

and take the limit $\kappa \to \infty$, $\partial_0 \to 0$, $\kappa\partial_0$ - finite. The result is ($t = \kappa\tau$)

$$\begin{aligned}
S &= \int d\tau d\sigma \mathcal{L}_{SC} \\
&= \frac{R^2\kappa}{2\lambda^0} \int dt d\sigma \left[\partial_0 \alpha + (r_1^2 - r_2^2)\partial_0 \varphi + r_3^2 \partial_0 \phi\right] \\
&- \frac{\lambda^0 (TR)^2}{\kappa} \int dt d\sigma \left\{\sum_{i=1}^{3} (\partial_1 r_i)^2\right. \\
&+ \left.\frac{4 r_1^2 r_2^2}{r_1^2 + r_2^2} (\partial_1 \varphi)^2 + (r_1^2 + r_2^2) r_3^2 \left[\left(\frac{r_1^2 - r_2^2}{r_1^2 + r_2^2}\right) \partial_1 \varphi - \partial_1 \phi\right]^2\right\} \\
&+ \frac{1}{\kappa} \int dt d\sigma \Lambda_S \left(\sum_{i=1}^{3} r_i^2 - 1\right).
\end{aligned}$$

The momentum P_α conjugated to α should be identified with the total angular momentum of the string J

$$P_\alpha = \frac{\pi R^2 \kappa}{\lambda^0} \equiv J.$$

Then the coefficients in the action become

$$\frac{R^2\kappa}{2\lambda^0} = \frac{J}{2\pi}, \quad \frac{\lambda^0(TR)^2}{\kappa} = \frac{\lambda}{4\pi J},$$

where we have used the relation $TR^2 = \sqrt{\lambda}/2\pi$ between the string tension T and the 't Hooft coupling λ.

If we parameterize the two-sphere in the following way

$$r_1 = \cos\psi\cos\theta, \quad r_2 = \sin\psi\cos\theta, \quad r_3 = \sin\theta,$$

the string action reduces to

$$\begin{aligned} S &= \frac{J}{2\pi}\int dtd\sigma\left[\partial_t\alpha + \cos^2\theta\cos(2\psi)\partial_t\varphi + \sin^2\theta\partial_t\phi\right] \\ &- \frac{\lambda}{4\pi J}\int dtd\sigma\left\{(\partial_\sigma\theta)^2 + \cos^2\theta\left[(\partial_\sigma\psi)^2 + \sin^2(2\psi)(\partial_\sigma\varphi)^2\right]\right. \\ &+ \left.\frac{1}{4}\sin^2(2\theta)\left[\cos(2\psi)\partial_\sigma\varphi - \partial_\sigma\phi\right]^2\right\}. \end{aligned}$$

This is the string action corresponding to the thermodynamic limit of $SU(3)$ spin chain after the identification $J \equiv L$ is made, where L is the length of the chain. The particular case of $SU(2)$ spin chain corresponds to $r_3 = 0$ or $\theta = 0$ in the above actions.

In order to make connection with the membrane case, let us fix $r_3^2 = \varepsilon^2$ and take the limit $\varepsilon^2 \to 0$. Neglecting the higher order terms, one obtains

$$\begin{aligned} S &= \frac{R^2\kappa}{2\lambda^0}\int d\tau d\sigma\left[\partial_0\alpha + (r_1^2 - r_2^2)\partial_0\varphi\right] \\ &- \lambda^0(TR)^2\int d\tau d\sigma\left\{\sum_{a=1}^{2}(\partial_1 r_a)^2\right. \\ &+ \left.\left[(r_1^2 + r_2^2) - (r_1^2 - r_2^2)^2\right](\partial_1\varphi)^2\right\} \\ &+ \int d\tau d\sigma\Lambda_S\left[\sum_{a=1}^{2} r_a^2 - (1-\varepsilon^2)\right]. \end{aligned}$$

According to the constraint in the above action, the coordinates r_1, r_2 must lie on a circle with radius $(1-\varepsilon^2)^{1/2}$. To satisfy this constraint identically, we choose

$$r_1 = (1-\varepsilon^2)^{1/2}\cos\psi, \quad r_2 = (1-\varepsilon^2)^{1/2}\sin\psi,$$

and receive ($\alpha = (1-\varepsilon^2)\tilde{\alpha}$)

$$\begin{aligned} S/(1-\varepsilon^2) &= \frac{R^2\kappa}{2\lambda^0}\int dtd\sigma\left[\partial_t\tilde{\alpha} + \cos(2\psi)\partial_t\varphi\right] \\ &- \frac{\lambda^0(TR)^2}{\kappa}\int dtd\sigma\left[(\partial_\sigma\psi)^2 + \sin^2(2\psi)(\partial_\sigma\varphi)^2\right], \end{aligned}$$

The right hand side coincides with the string action corresponding to the thermodynamic limit of the $SU(2)$ spin chain action.

3.2 MEMBRANES ON $AdS_4 \times S^7$

For membranes we will use our initial embedding and fix

$$Z_0 = 2l_p\mathcal{R}e^{i\kappa\tau}, \quad Z_1 = 0,$$

which implies

$$\mathrm{r}_0 = 1, \quad \mathrm{r}_1 = 0, \quad \phi_0 = t = \kappa\tau.$$

Let us now introduce new coordinates by setting

$$\begin{aligned} &(\varphi_1, \varphi_2, \varphi_3, \varphi_4) \\ &= \left(\frac{\kappa}{2}\tau + \alpha + \varphi, \frac{\kappa}{2}\tau + \alpha - \varphi, \frac{\kappa}{2}\tau + \alpha + \phi, \frac{\kappa}{2}\tau + \alpha - \tilde{\phi}\right) \end{aligned}$$

and take the limit $\kappa \to \infty$, $\partial_0 \to 0$, $\kappa\partial_0$ - finite. In this limit, we obtain the following expression for the membrane Lagrangian

$$
\begin{aligned}
\mathcal{L} &= \frac{(2l_p\mathcal{R})^2}{\lambda^0}\kappa\left(\partial_0\alpha + \sum_{k=1}^{3}\nu_k\partial_0\rho_k\right) \\
&- \lambda^0 T_2^2(4l_p\mathcal{R})^4\left\{\sum_{a<b=1}^{4}(\partial_1 r_a\partial_2 r_b - \partial_2 r_a\partial_1 r_b)^2\right. \\
&+ \sum_{a=1}^{4}\sum_{k=1}^{3}\mu_k(\partial_1 r_a\partial_2\rho_k - \partial_2 r_a\partial_1\rho_k)^2 \\
&- \sum_{a=1}^{4}\left(\partial_1 r_a\sum_{k=1}^{3}\nu_k\partial_2\rho_k - \partial_2 r_a\sum_{k=1}^{3}\nu_k\partial_1\rho_k\right)^2 \\
&+ \sum_{k<n=1}^{3}\mu_k\mu_n(\partial_1\rho_k\partial_2\rho_n - \partial_2\rho_k\partial_1\rho_n)^2 \\
&- \left.\sum_{k=1}^{3}\mu_k\left(\partial_1\rho_k\sum_{n=1}^{3}\nu_n\partial_2\rho_n - \partial_2\rho_k\sum_{n=1}^{3}\nu_n\partial_1\rho_n\right)^2\right\} \\
&+ \Lambda_S\left(\sum_{a=1}^{4}r_a^2 - 1\right),
\end{aligned}
$$

where

$$
\begin{aligned}
(\mu_1,\mu_2,\mu_3) &= (r_1^2 + r_2^2, r_3^2, r_4^2), \\
(\nu_1,\nu_2,\nu_3) &= (r_1^2 - r_2^2, r_3^2, -r_4^2), \\
(\rho_1,\rho_2,\rho_3) &= (\varphi,\phi,\tilde{\phi}).
\end{aligned}
$$

Now, we are ready to face our main problem: how to reduce this Lagrangian to the one corresponding to the thermodynamic limit of spin chain, *without shrinking the membrane to string*? We propose the following solution of this task:

$$
\begin{aligned}
&\alpha = \alpha(\tau,\sigma_1), \quad r_1 = r_1(\tau,\sigma_1), \quad r_2 = r_2(\tau,\sigma_1), \\
&r_3 = r_3(\tau,\sigma_2) = a\sin[b\sigma_2 + c(\tau)], \\
&r_4 = r_4(\tau,\sigma_2) = a\cos[b\sigma_2 + c(\tau)], \\
&\varphi = \varphi(\tau,\sigma_1), \quad a, b, \phi, \tilde{\phi} = constants, \quad a^2 < 1.
\end{aligned}
$$

These restrictions lead to

$$\begin{aligned}
\mathcal{L} &= \frac{(2l_p\mathcal{R})^2}{\lambda^0}\kappa\left[\partial_0\alpha + (r_1^2 - r_2^2)\partial_0\varphi\right] \\
&- \lambda^0(abT_2)^2(4l_p\mathcal{R})^4\left\{\sum_{a=1}^{2}(\partial_1 r_a)^2\right. \\
&+ \left.\left[(r_1^2 + r_2^2) - (r_1^2 - r_2^2)^2\right](\partial_1\varphi)^2\right\} \\
&+ \Lambda_S\left[\sum_{a=1}^{2} r_a^2 - (1 - a^2)\right].
\end{aligned}$$

The above membrane Lagrangian is fully analogous to the string Lagrangian, obtained after fixing r_3^2 to $\varepsilon^2 \to 0$. Proceeding as in the string case, we introduce the parametrization

$$r_1 = (1 - a^2)^{1/2}\cos\psi, \quad r_2 = (1 - a^2)^{1/2}\sin\psi,$$

the new variable $\tilde{\alpha}$

$$\alpha = (1 - a^2)\tilde{\alpha},$$

and take the limit $a^2 \to 0$. Thus, we receive

$$\begin{aligned}
\mathcal{L}/(1 - a^2) &= \frac{(2l_p\mathcal{R})^2}{\lambda^0}\kappa\left[\partial_0\tilde{\alpha} + \cos(2\psi)\partial_0\varphi\right] \\
&-\lambda^0(abT_2)^2(4l_p\mathcal{R})^4\left[(\partial_1\psi)^2 + \sin^2(2\psi)(\partial_1\varphi)^2\right]
\end{aligned}$$

As for the membrane action corresponding to the above Lagrangian, it can be represented in the form

$$\begin{aligned}
S_M &= \frac{\mathcal{J}}{2\pi}\int dtd\sigma\left[\partial_t\tilde{\alpha} + \cos(2\psi)\partial_t\varphi\right] \\
&- \frac{\tilde{\lambda}}{4\pi\mathcal{J}}\int dtd\sigma\left[(\partial_\sigma\psi)^2 + \sin^2(2\psi)(\partial_\sigma\varphi)^2\right],
\end{aligned}$$

where $\mathcal{J}$ is the angular momentum conjugated to $\tilde{\alpha}$, $t = \kappa\tau$ and

$$\tilde{\lambda} = 2^{15}[\pi^2(1 - a^2)abT_2]^2(l_p\mathcal{R})^6.$$

Obviously, this action corresponds to the thermodynamic limit of $SU(2)$ integrable spin chain.

CONCLUSION

THERE SHOULD EXIST
COMMON INTEGRABLE SECTORS
IN
$\mathcal{N} = 4$ SYM IN FOUR DIMENSIONS
AND
IN
THE THREE DIMENSIONAL
CONFORMAL FIELD THEORY
DUAL TO MEMBRANES ON $AdS_4 \times S^7$,
ACCORDING TO THE MALDACENA CONJECTURE !