Supersymmetric Black Holes: Special Geometry, Partition Functions, Instantons and the r-map

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Outline

Part I Black hole entropy and black hole partition functions.

Based on joined work with Bernard de Wit, Gabriel Lopes Cardoso and Jürg Käppeli, i.p. `JHEP 03 (2006) 074`.

Review: `T.M., Fortschr. Phys. 55 (2007) 519 .`

- Review of black hole entropy, BPS (supersymmetric) black holes and the string – black hole correspondence.
- $\mathcal{N} = 2$ compactifications, special geometry and the attractor mechanism.
- Variational principles and black hole partition functions, the OSV conjeture.
- Modified (duality covariant) version of the OSV conjecture, tests in $\mathcal{N} = 4$ compactifications.

Part II From black holes to instantons: special geometry for Euclidean Supergravity and the r-map.

Based on work to appear/in progress with Vicenté Cortes.

The Laws of Black Hole Mechanics

(0) $\kappa_S =$ const.

(1) $\delta M = \frac{\kappa_S}{8\pi}\delta A + \omega \delta J + \phi \delta q$.

(2) $\delta A \geq 0$.

(3) $\kappa_S = 0$ cannot be reached in finite time by any physical process.

$ks =$ surface gravity, $M =$ mass, $A =$ horizon area, $\omega =$ horizon angular velocity, $J =$ angular momentum, $\phi =$ chemical (= electrostatic) potential, $q =$ electric charge.

`J.M. Bardeen, B. Carter and S.W. Hawking (1973)`

Suggests:

$$\kappa_S \sim T\,, \quad A \sim \mathrm{S}$$

where $T =$ temperature and $\mathrm{S} =$ Entropy.

The ‘classical’ proofs of these theorems uses Einstein’s field equations explicitly (together with other assumptions).

However, (0), (1) hold irrespective of the details of the field equations, if appropriate symmetry conditions are imposed!

The Laws of Black Hole Mechanics (2)

Modified assumptions: generally covariant Lagrangian with black hole solution such that

(i) black hole is static or stationary, axisymmetric, $t - \phi$ reflection symmetric,

(ii) event horizon is a Killing horizon,

(iii) a Cauchy hypersurface exists.

Then

(0) $\kappa_S =$ const.
`R. Wald and Racz (1995)`

(1) $\delta M = \frac{\kappa_S}{2\pi} \delta \mathrm{S} + \omega \delta J + \phi \delta q$.
`R. Wald (1993), ...`

provided that M, J, S, q are defined as appropriate surface charges.

Entropy:

$$\mathrm{S} = \int_H Q[\xi]$$

$\xi =$ 'horizontal' Killing vector field, $Q =$ Noether two-form.

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Entropy:

$$\mathrm{S} = 2\pi \int_H \frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \varepsilon^{\mu\nu} \varepsilon^{\rho\sigma} \sqrt{h} d^2\Omega$$

$\varepsilon^{\mu\nu} =$ normal bivector of horizon, $\sqrt{h} d^2\Omega =$ induced volume form.

Black Hole Thermodynamics

Hawking radiation:

$$T_{\mathrm{Hawking}} = \frac{\kappa_S}{2\pi} \quad (G_N = c = \hbar = 1)\,.$$

First law:

$$\delta M = \frac{\kappa_S}{8\pi}\delta A + \cdots$$

$$\Rightarrow \mathrm{S} = \frac{A}{4}\,.$$

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$\mathrm{S} = \mathrm{S}_{\text{macro}} =$ thermodynamical or macroscopic entropy. Unterlying microscopic theory (=quantum gravity) should specify the microscopic states of the black hole.

Microscopic entropy:

$$\mathrm{S}_{\text{micro}} = \log d(M, J, q)\ , \quad d = \#\text{microstates}\ .$$

Expect: $\mathrm{S}_{\text{macro}} = \mathrm{S}_{\text{micro}}\ .$

Benchmark for theories of quantum gravity!

The String – Black Hole Correspondence

Idea: black hole microstates $=$ string states at large mass or large coupling.

Semiclassical gravity regime.	$\leftrightarrow$	Perturbative string regime
$\sqrt{\alpha'} \ll r_S$.		$\sqrt{\alpha'} \gg r_S$

`L. Susskind (1993), G.T. Horowitz and J. Polchinski (1997).`

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Compare 4d Schwarzschild black hole to open bosonic string (truncated).
Relation of gravitational and string scale: $G_N = g_S^2 \alpha'$.
String mass formula:

$$\alpha' M^2 \approx n \quad \text{(for large excitation number } n\text{)} \ .$$

String entropy:

$$\mathrm{S}_{\mathrm{String}} = \log d(n) \sim \sqrt{n} \ .$$

Schwarzschild radius of string state:

$$r_S \approx g_S^2 \sqrt{n} \sqrt{\alpha'} \ .$$

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- Perturbative string in flat space:

$$g_S^2 \sqrt{n} \ll 1 \Rightarrow r_S \ll \sqrt{\alpha'} \,, \quad \mathrm{S}_{BH} \ll \mathrm{S}_{\mathrm{String}}$$

- Semiclassical black hole:

$$g_S^2 \sqrt{n} \gg 1 \Rightarrow r_S \gg \sqrt{\alpha'} \,, \quad \mathrm{S}_{BH} \gg \mathrm{S}_{\mathrm{String}} \,.$$

- Transition/Crossover(?):

$$g_S^2 \sqrt{n} \approx 1 \Rightarrow r_S \approx \sqrt{\alpha'} \,, \quad \mathrm{S}_{BH} \approx \mathrm{S}_{\mathrm{String}} \,.$$

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$\sqrt{\alpha'} \ll r_S$.		$\sqrt{\alpha'} \gg r_S$

Need to interpolate between two accessible regimes. Intermediate regime not under control.
Matching of entropies up to $\mathcal{O}(1)$:

$$\mathrm{S}_{\mathrm{BH}} \sim \mathrm{S}_{\mathrm{String}} \ .$$

Consider supersymmetric (BPS) states. `A. Strominger and C. Vafa (1996), ...`
Interpolation more plausible.
Can compute $\mathrm{S}_{\mathrm{macro}} \equiv \mathrm{S}_{\mathrm{BH}}$ and $\mathrm{S}_{\mathrm{micro}} \equiv \mathrm{S}_{\mathrm{String}}$ to high precision in their respective regimes.
Find 'exact' matching

$$\mathrm{S}_{\mathrm{BH}} \approx \mathrm{S}_{\mathrm{String}} \ ,$$

including subleading corrections (in large mass $=$ semiclassical expansion).

The String – Black Hole Correspondence

Idea: black hole microstates $=$ string states at large mass or large coupling.

Semiclassical gravity regime.	$\leftrightarrow$	Perturbative string regime
$\sqrt{\alpha'} \ll r_S$.		$\sqrt{\alpha'} \gg r_S$

Besides fundamental strings, also solitonic p-branes are associated with microscopic degrees of freedom.

E.g. the pioneering work of A. Strominger and C. Vafa involved D-branes, rather than fundamental strings.

We will discuss examples involving fundamental (and also solitonic) strings later.

BPS states

Supersymmetry algebra (4d, Weyl spinors):

$$\{Q_\alpha, Q^+_\beta\} = 2\sigma^\mu_{\alpha\beta} P_\mu$$

BPS states

$\mathcal{N}$-extended supersymmetry algebra (4d, Weyl spinors):

$$\begin{aligned}\{Q_\alpha^A, Q_\beta^{+B}\} &= 2\sigma^\mu_{\alpha\beta}\delta^{AB}P_\mu \\ \{Q_\alpha^A, Q_\beta^B\} &= \epsilon_{\alpha\beta}Z^{AB}\end{aligned}$$

$A, B, \ldots = 1, \ldots, N$.
Central charges $=$ skew eigenvalues of Z^{AB}:

$$M^2 \geq |Z_1|^2 \geq |Z_2|^2 \geq \cdots \geq 0 \ .$$

Saturation of inequalities $\Rightarrow$ shortened (BPS) multiplets.

BPS states

$\mathcal{N}$-extended supersymmetry algebra (4d, Weyl spinors):

$$\begin{aligned} \{Q^A_\alpha, Q^{+B}_\beta\} &= 2\sigma^\mu_{\alpha\beta}\delta^{AB}P_\mu \\ \{Q^A_\alpha, Q^B_\beta\} &= \epsilon_{\alpha\beta}Z^{AB} \end{aligned}$$

$A, B, \ldots = 1, \ldots, N$.

Examples:

- $\mathcal{N} = 2$:
  1. $M > |Z|$: generic massive multiplet.
  2. $M = |Z|$: short or $\frac{1}{2}$-BPS multiplet.
- $\mathcal{N} = 4$:
  1. $M > |Z_1| > |Z_2|$: generic massive multiplet.
  2. $M = |Z_1| > |Z_2|$: $\frac{1}{4}$-BPS multiplet.
  3. $M = |Z_1| = |Z_2|$: $\frac{1}{2}$-BPS multiplet.

Supersymmetric vacua $=$ '$\frac{1}{1}$-BPS' (fully supersymmetric).

BPS solitons

BPS states can be realized as finite energy solutions Φ_0 of the field equations (asymptotic to vacuum).

Killing spinors $\varepsilon \leftrightarrow$ residual (rigid) supersymmetry of Φ_0:

$$\delta_\varepsilon \ \Phi|_{\Phi_0} = 0 \ .$$

Example: the extremal Reissner-Nordstrom black hole regarded as a solution of $\mathcal{N} = 2$ Supergravity = Einstein-Maxwell + 2 Gravitini.

Has 4 Killing spinors and interpolates between two supersymmetric vacua (8 Killing spinors): Minkowski space at infinity and $AdS^2 \times S^2$ (with covariantly constant gauge fields) at horizon.

`G. Gibbons (1982)`

Embedding into string theory

String compactifications give supergravity plus matter.

We consider:

- Het$/(K3 \times T^2)$ and type-II/CY_3
  $\longrightarrow \mathcal{N} = 2$ Supergravity + n_V vector multiplets (+ n_H hypermultiplets + n_T tensor multiplets).
- Het$/T^6$ and type-II/$(K3 \times T^2)$
  $\longrightarrow \mathcal{N} = 4$ Supergravity + n_V vector multiplets.

Main tool: special geometry of $\mathcal{N} = 2$ vector multiplets.
`B. de Wit and A. Van Proeyen (1984).`

All vector multiplet couplings are encoded in a holomorphic prepotential.
Field equations are invariant under $Sp(2n_V + 2, \mathbb{R})$ rotations which generalize the electric-magnetic duality of Maxwell theory, and include stringy symmetries such as T-duality and S-duality.

Special geometry (1)

Multiplets:

- Gravity multiplet: $\{e_\mu^{\ A}, \psi_\mu^i, \mathcal{A}_\mu\}$.
- Vector multiplet: $\{\mathcal{A}_\mu, \lambda^i, z\}^A$.

$i = 1, 2$, $A = 1, \ldots, n_V$.

Bosonic Lagrangian:

$$8\pi e^{-1}\mathcal{L}_{\rm bos} = -\frac{R}{2} - g_{A\overline{B}}(z,\overline{z})\partial_\mu z^A \partial^\mu \overline{z}^{\overline{B}} + \frac{i}{4}\overline{\mathcal{N}}_{IJ}(z,\overline{z})F^{-I}_{\mu\nu}F^{-I|\mu\nu} - \frac{i}{4}\mathcal{N}_{IJ}(z,\overline{z})F^{+I}_{\mu\nu}F^{+I|\mu\nu}$$

$I = 0, 1, \ldots, n_V$.
$F^{\pm I}_{\mu\nu}$ = (anti-)selfdual part of field strength.

To make electric-magnetic duality manifest, define dual field strength:

$$G^{\pm}_{I|\mu\nu} \simeq \frac{\delta \mathcal{L}}{\delta F^{\pm I|\mu\nu}} \ .$$

Special Geometry (2)

Field equations are invariant under $Sp(2n_V + 2, \mathbb{R})$ rotations.

Symplectic vectors:

- Gauge fields and charges: $\begin{pmatrix} F^{\pm I}_{\mu\nu} \\ G^{\pm}_{I|\mu\nu} \end{pmatrix}$, $\begin{pmatrix} p^I \\ q_I \end{pmatrix} = \begin{pmatrix} \oint F^I \\ \oint G_I \end{pmatrix}$
- Scalars: $\begin{pmatrix} X^I \\ F_I \end{pmatrix}$

where 'scalars' X^I are related to the physical scalars z^A by $z^A = \frac{X^A}{X^0}$ and

$$F_I = \frac{\partial F}{\partial X^I} \ .$$

Prepotential $F(X)$ is holomorphic and homogenous of degree 2:

$$F(\lambda X^I) = \lambda^2 F(X) \ .$$

Special Geometry (3)

Poincaré Supergravity	$\longleftrightarrow$	Conformal Supergravity		
n_V vector multiplets		$(n_V + 1)$ vector multiplets		
$\mathcal{M}_{VM}$	$\longleftrightarrow$	$\mathcal{CM}_{VM}$	$\xrightarrow{\Phi}$	$\mathbb{C}^{2n_V+2}$
z^A		X^I		$\begin{pmatrix} X^I \\ F_I \end{pmatrix}$

$\mathcal{CM}_{VM}$ is a complex cone over $\mathcal{M}_{VM}$. (Prepotential needs to be homogenous).

(Holomorphic) Prepotential F defines holomorphic Lagrangian immersion, $\Phi = dF$ into symplectic vector space $\mathbb{C}^{2n_V+2} = T^*\mathbb{C}^{n_V+1}$.

`Embedding construction explained in: D.V. Alekseevsky, V. Cortés and C. Devchand, math.dg/9910091.`

Special Geometry (3)

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Special geometry is naturally realised in type-II Calabi-Yau compactifications:

$\mathcal{M}_{VM}$ = complex structure moduli (IIB).
$\mathcal{CM}_{VM}$ = complex structure moduli + holomorphic volume form.
$\mathbb{C}^{2n_V+2} = H^3(X, \mathbb{C})$. Middle cohomology.

Special Geometry (4)

Real parametrizations and Hesse potential.

Kähler manifold: metric, complex and sympletic structure mutually compatible.
Special Kähler manifold: existence of a flat, torsion-free, symplectic connection, such that $\nabla_{[\mu} I^{\nu}{}_{\rho]} = 0$. D.S. Freed (1997)

Special real coordinates:

$$x^I = \mathrm{Re} X^I \; , \quad y_I = \mathrm{Re} F_I \; .$$

Legendre transform $X^I = (\mathrm{Re}(X^I), \mathrm{Im}(X^I)) \to (x^I, y_I)$. All couplings encoded in the Hesse potential (real Kähler potential):

$$H(x^I, y_I) = 2\left(\mathrm{Im}F - (\mathrm{Re}F_I)(\mathrm{Im}X^I)\right) \quad \left(\text{note } \; \mathrm{Re}F_I = \frac{\partial(\mathrm{Im}F)}{\partial(\mathrm{Im}X^I)}\right) .$$

V. Cortés (2001).

The real parametrization is very useful for BPS black holes and other BPS solitons.

BPS Black Holes

Impose

- $\frac{1}{2}$-BPS solution, i.e. 4 linearly independent Killing spinors ε_i: $\delta_{\varepsilon_i}\Phi = 0$.
  $\Phi =$ all fields.
- Solution static and spherically symmetric.
  Metric:

$$ds^2 = -e^{2g(r)}dt^2 + e^{2f(r)}(dr^2 + r^2 d\Omega^2)$$

$\frac{1}{2}$-BPS $\Rightarrow g(r) = -f(r)$.
Gauge fields (using orthonormal frame):

$$F^I_{\underline{tr}} = F^I_E(r) \sim q_I \ , \quad F^I_{\underline{\theta\phi}} = F^I_M(r) \sim p^I$$

Scalars: $z^A = z^A(r) = \frac{X^A(r)}{X^0(r)} = \frac{Y^A(r)}{Y^0(r)}$

$X^I, Y^I = \overline{Z}X^I$ are related by an (r-dependent) rescaling.
$Z = p^I F_I - q_I X^I =$ ‘central charge.’

New features, compared to extreme Reissner-Nordstrom black hole: several gauge fields and charges, scalars.

Attractor mechanism

1. $r \longrightarrow \infty$: solution asymptotically flat.
   $z^A \longrightarrow z^A(\infty) \in \mathcal{M}_{VM}$.
   Minkowski vacuum with arbitrary moduli.
2. $r \longrightarrow 0$ (event horizon): $z^A \longrightarrow z^A_*(p,q)$.
   Gauge fields and metric determined by $z^A_*(p,q)$:
   Solution is Bertotti-Robinson solution $AdS^2 \times S^2$ with radius fixed by the charges.
   Horizon solution = supersymmetric vacuum with fixed moduli.

Attractor mechanism! `S. Ferrara, R. Kallosh and A. Strominger (1995)`.

Attractor equations:

$$\left(\begin{array}{c} Y^I - \overline{Y}^I \\ F_I - \overline{F}_I \end{array} \right)_* = i \left(\begin{array}{c} p^I \\ q_I \end{array} \right)$$

Entropy:

$$\mathrm{S}_{\mathrm{macro}} = \frac{A}{4} = \pi |Z|^2_* = \pi |p^I F_I(X) - q_I X^I|^2_* = \pi \left(p^I F_I(Y) - q_I Y^I \right)_*$$

Variational Principle

Complex version.

Entropy function:

$$\Sigma(Y,\overline{Y},p,q) := \mathcal{F}(Y,\overline{Y}) - q_I(Y^I + \overline{Y}^I) + p^I(F_I + \overline{F}_I) \; .$$

Free energy:

$$\mathcal{F}(Y,\overline{Y}) = -i(F_I \overline{Y}^I - Y^I \overline{F}_I) \; .$$

Critical points:

$$\frac{\partial \Sigma}{\partial Y^I} = 0 = \frac{\partial \Sigma}{\partial \overline{Y}^I} \Longleftrightarrow \quad \text{attractor equations}$$

Critical value $\propto$ entropy:

$$\pi \Sigma_* = \mathrm{S}_{\mathrm{macro}}(p,q) \; .$$

K. Behrndt, G.L. Cardoso, B. de Wit, R. Kallosh, D. Lüst and T.M. (1996)

Variational Principle

Real version:

Entropy function:

$$\Sigma(x,y,q,p) = 2H(x,y) - 2q_I x^I + 2p^I y_I \ .$$

Free energy $\propto$ Hesse potential:

$$2H(x,y) = \mathcal{F}(Y,\overline{Y}) = -i(F_I \overline{Y}^I - Y^I \overline{F}_I) \ .$$

Critical points:

$$\frac{\partial H}{\partial x^I} = q_I \ , \quad \frac{\partial H}{\partial y_I} = -p^I \Leftrightarrow \text{ attractor equations .}$$

Entropy $=$ (full) Legendre transform of Hesse potential:

$$\mathrm{S}_{\mathrm{macro}}(p,q) = 2\pi \left(H - x^I \frac{\partial H}{\partial x^I} - y_I \frac{\partial H}{\partial y_I} \right)_*$$

`G.L. Cardoso, B. de Wit, J. Käppeli and T.M., JHEP 03 (2006) 074, hep-th/0601108.`

R^2–Corrections

String-effective actions contain an infinite series of higher-derivative terms, computable (in principle) from string perturbation theory.

In $\mathcal{N}=2$ supergravity, there is an off-shell description for a particular class of such terms, (called 'R^2-terms').

Conformal Supergravity: gravitational degrees of freedom reside in the Weyl multiplet.

Generalized prepotential with explicit dependence on the Weyl multiplet:

$$F(Y^I) \rightarrow F(Y^I, \Upsilon) ,$$

where $\Upsilon =$ lowest component of Weyl multiplet $\sim$ Graviphoton field strength.

$F(Y^I, \Upsilon)$ is holomorphic and (graded) homogenous of degree two:

$$F(\lambda Y^I, \lambda^2 \Upsilon) = \lambda^2 F(Y^I, \Upsilon) .$$

Weyl multiplet can be treated as a background field.
`B. de Wit, hep-th/9602060, 9603191`.

R^2–Corrections (2)

Expand in Υ:

$$F(Y^I, \Upsilon) = \sum_{g=0}^{\infty} F^{(g)}(Y^I)\Upsilon^g \ .$$

Higher derivative terms include:

$$\mathcal{L} \sim \sum_{g=1}^{\infty} \Big(F^{(g)}(Y^I)(C^-_{\mu\nu\rho\sigma})^2 (T^-_{\alpha\beta})^{2g-2} + \text{ c.c. } \Big) + \cdots$$

$C_{\mu\nu\rho\sigma}$ = Weyl tensor, $T_{\mu\nu}$ = Graviphoton field strength.

In type-II CY compactifications, $F^{(g)}(Y^I)$ can be computed in the topologically twisted theory.
$F^{(g)} \propto$ genus g contribution to topological free energy, i.e. $\exp(F^{(g)}) \propto$ partition function of topological string on genus–g world sheet.

M. Beshadsky, S. Cecotti and H. Ooguri (1993).
I. Antoniadis, E. Gava, C. Narain and T.R.Taylor (1993).

R^2–Corrections (3)

Attractor mechanism generalizes.
Solutions can be constructed, at least iteratively.

Attractor equations:

$$\begin{pmatrix} Y^I - \overline{Y}^I \\ F_I(Y,\Upsilon) - \overline{F}_I(\overline{Y},\overline{\Upsilon}) \end{pmatrix}_* = i \begin{pmatrix} p^I \\ q_I \end{pmatrix} , \quad \Upsilon_* = -64 \; .$$

Entropy:

$$\mathrm{S}_{\mathrm{macro}} = \mathrm{S}_{\mathrm{Wald}} = \pi \left((p^I F_I(Y,\Upsilon) - q_I Y^I) - 256 \,\mathrm{Im} \left(\frac{\partial F}{\partial \Upsilon} \right) \right)_*$$

Note: modification of area and modification of area law.
Important for matching $\mathrm{S}_{\mathrm{macro}} = \mathrm{S}_{\mathrm{micro}}$.

`G.L. Cardoso, B. de Wit and T.M. (1998), G.L. Cardoso, B. de Wit, J. Käppeli and T.M. (2000)`

Reduced variational principle

Canonical ensemble. Entropy function:

$$\begin{aligned}\Sigma(x,y,q,p) &= 2H(x,y) - 2q_I x^I + 2p^I y_I \\ = \Sigma(Y,\overline{Y},p,q) &= \mathcal{F}(Y,\overline{Y}) - q_I(Y^I + \overline{Y}^I) + p^I(F_I + \overline{F}_I)\end{aligned}$$

$(x^I, y_I) \propto (\phi^I, \chi_I)$ electro-/magneto-static potentials = chemical potentials. Related to electric/magnetic charges by Legendre transform.

Mixed ensemble. Solve magnetic attractor equations:

$$Y^I - \overline{Y}^I = ip^I \Rightarrow Y^I = \frac{1}{2}(\phi^I + ip^I)$$

and substitute into Σ. Partial Legendre transform $\chi_I \to p^I$. Reduced entropy function (used by OSV):

$$\Sigma(\phi,p,q)_{\mathrm{mix}} = \mathcal{F}_{\mathrm{mix}}(p,\phi) - q_I\phi^I$$

where $\mathcal{F}_{\mathrm{mix}}(p,\phi) = 4\mathrm{Im}F(Y,\overline{Y})$

(we suppress Υ in the following, but R^2-corrections are included).

Reduced variational principle (2)

Variation of the reduced entropy function:

$$\Sigma(\phi, p, q)_{\rm mix} = \mathcal{F}_{\rm mix}(p, \phi) - q_I \phi^I$$

where $\mathcal{F}_{\rm mix}(p, \phi) = 4\text{Im}F(Y, \overline{Y})$ yields the electric attractor equations:

$$\frac{\partial \mathcal{F}_{\rm mix}}{\partial \phi^I} = q_I$$

Moreover: black hole entropy = partial Legendre transform ($ReY^I \propto x^I \propto \phi^I \to q_I$) of free energy $\mathcal{F}_{\rm mix}$:

$$\mathrm{S}_{\rm macro}(p, q) = \pi \Sigma_* = \pi \left(\mathcal{F}_{\rm mix} - \phi^I \frac{\partial \mathcal{F}_{\rm mix}}{\partial \phi^I} \right)_*$$

OSV conjecture

Observation:

$$e^{\pi \mathcal{F}_{\text{mix}}} = |e^{F_{\text{top}}}|^2 = |Z_{\text{top}}|^2$$

where $F_{\text{top}} \propto iF(Y^I, \Upsilon)$ is the (properly normalized) all-genus free energy of the topological type-II string, and $Z_{\text{top}} =$ all-genus topological partition function.

This is ‘just’ the relation between the topological string and couplings in the effective action.

Conjecture (`H. Ooguri, A. Strominger and C. Vafa (2004)`):

$$e^{\pi \mathcal{F}_{\text{mix}}} \approx Z_{BH}(p, \phi)$$

should be interpreted as the partition function of the black hole (in the mixed ensemble),

$$Z_{BH}(p, \phi) = \sum_q d(p, q) e^{\pi q_I \phi^I}$$

$d(p, q)$: microscopic degeneracy (microcanonical ensemble: electric and magnetic charges fixed).
Ensembles are related by (discrete) Laplace transforms.

OSV conjecture (2)

Inverse Laplace transform gives prediction of state degeneracy:

$$d(p,q) \approx \int_C d\phi^I \; e^{\pi \mathcal{F}_{\rm mix} - q_I \phi^I} \; .$$

Relation between partition functions:

$$Z_{BH} \approx |Z_{\rm top}|^2 \; .$$

Weak version: this holds asymptotically in the semiclassical limit = limit of large charges. E.g. to all orders in $1/Q$, $Q =$ charges.
Successfully tested for ‘large’ black holes, not so clear for ‘small’ black holes.

Strong version: exact statement (at least once appropriate amendments are made, i.p. giving up holomorphic factorization, see later). Intriguing, but much less clear.

Tests: predict free energy $\mathcal{F}$ from microscopic state degeneracy $d(p,q)$, or vice versa.
Besides matching numbers (expansion coefficients), try to match structures in Z_{BH} and $Z_{\rm top}$ (modular forms, Rademacher-type expansions). I.p. explain $Z_{BH} \approx |Z_{\rm top}|^2$.

Nonholomorphic Corrections

- The topological string has a holomorphic anomaly.
  `M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa (1993), E. Witten (1993),`
  Consequences for OSV have been discussed from this point of view by
  `R. Dijkgraaf, S. Gukov, A. Neitzke and C. Vafa (2005), E. Verlinde (2005), ...`
- Effective field theory: Wilsonian effective action (local, IR cut-off) vs. generating functional of 1PI graphs (non-local i.g, when massless modes are present). Physical, duality covariant couplings include non-holomorphic corrections.
  `L. Dixon, V. Kaplunovski and J. Louis (1991)`
- The same applies to the entropy of BPS black holes. Generalization of attractor equations and variational principle:

$$\text{Im}(F(Y,\Upsilon)) \to \text{Im}(F(Y,\Upsilon)) + 2\Omega(Y,\overline{Y},\Upsilon,\overline{\Upsilon})$$

  where Ω is real-valued, homogenous of degree 2 and (i.g.) not harmonic.
  `G.L. Cardoso, B. de Wit and T.M. (1999), G.L. Cardoso, B. de Wit, J. Käppeli and T.M. (2006).`

A modified version of OSV

Macroscopic entropy = full Legendre transform of Hesse potential. This suggests:

$$\begin{aligned} e^{2\pi H(x,y)} &\approx Z_{\mathrm{BH}}^{(\mathrm{can})} = \sum_{p,q} d(p,q) e^{2\pi[q_I x^I - p^I y_I]} \\ \Leftrightarrow e^{\pi \mathcal{F}(Y,\overline{Y})} &\approx Z_{\mathrm{BH}}^{(\mathrm{can})} = \sum_{p,q} d(p,q) e^{\pi[q_I(Y^I+\overline{Y}^I) - p^I(F_I+\overline{F}_I)]} \end{aligned}$$

(we suppressed Υ, but inclusion of R^2 and nonholomorphic corrections is understood.)

- Canonical rather than mixed ensemble.
- Symplectic covariance manifest.
- Nonholomorphic corrections are included ab initio.
- So far, we only have evidence for the weak form of the conjecture.

G.L. Cardoso, B. de Wit, J. Käppeli and T.M. (2006).

A modified version of OSV (2)

Formal inverse Laplace transform gives state degeneracy in terms of black hole free energy:

$$d(p,q) \approx \int dx dy e^{\pi\Sigma(x,y)} \approx \int dY d\overline{Y} \Delta^-(Y,\overline{Y}) e^{\pi\Sigma(Y,\overline{Y})}$$

where

$$\Delta^{\pm}(Y,\overline{Y}) = |\det\left[\mathrm{Im}F_{KL} + 2\mathrm{Re}(\Omega_{KL} \pm \Omega_{K\overline{L}})\right]|$$

$\Delta^- =$ measure factor, $\Delta^+ =$ fluctuation determinant of saddle point integral.

Microscopic entropy:

$$d(p,q) = e^{\mathrm{S}_{\mathrm{micro}}(p,q)}$$

Macroscopic entropy, in saddle point approximation:

$$d(p,q) \approx e^{\mathrm{S}_{\mathrm{macro}}(p,q)} \sqrt{\frac{\Delta^-}{\Delta^+}} \approx e^{\mathrm{S}_{\mathrm{macro}}(p,q)(1+\cdots)}$$

Recall: $\mathrm{S}_{\mathrm{macro}}(p,q) = \pi\Sigma_*$. Note: $\Delta^{\pm}$ are subleading.

Comparison to OSV

Partial saddle point approximation wrt ImY^I $\Leftrightarrow$ Imposing magnetic attractor equations. (Electric attractor equations follow from extremization of reduced entropy function.)

$$d(p,q) \approx \int d\phi \sqrt{\Delta^-(p,\phi)}\, e^{\pi[\mathcal{F}_{\rm mix}(\phi,p) - q_I \phi^I]}$$

$\mathcal{F}_{\rm mix}(\phi,p) =$ free energy in mixed ensemble, including the nonholomorphic corrections encoded in Ω!

Invert and compare to OSV:

$$\begin{aligned} \sqrt{\Delta^-}\, e^{\pi \mathcal{F}_{\rm mix}(p,\phi)} &\approx Z_{\rm BH}^{(\rm mix)} = \sum_q d(p,q) e^{\pi q_I \phi^I} \\ e^{\pi \mathcal{F}_{OSV}(p,\phi)} &\approx Z_{\rm BH}^{(\rm mix)} = \sum_q d(p,q) e^{\pi q_I \phi^I} \end{aligned}$$

Original OSV conjecture does not have the measure factor Δ^- (which is implied by symplectic covariance) and $\mathcal{F}_{OSV}$ does not include non-holomorphic corrections in the free energy.

State counting shows that these additional terms are indeed present.

BPS states in $\mathcal{N} = 4$ compactifications

Charges: $(p, q) \in \Gamma_{\mathrm{magnetic}} \oplus \Gamma_{\mathrm{electric}}$.

1. $p^2 q^2 - (p \cdot q)^2 = 0$.
   Short or $\frac{1}{2}$-BPS multiplet. Example of realisation: fundamental heterotic string states in Het$/T^6$, $p = 0$.
   Correspond to 'small' black holes (null singularity at two-derivative level, resolved by R^2-corrections, `A. Dabholkar, R. Kallosh and A. Maloney (2004).`).

$$\mathcal{S}_{\mathrm{macro}} = 0 + 4\pi \sqrt{\frac{1}{2}|q^2|} + \cdots$$

2. $p^2 q^2 - (p \cdot q)^2 \neq 0$.
   Intermediate or $\frac{1}{4}$-BPS multiplet. Example of realisation: bound state of fundamental string and solitonic five-brane in Het$/T^6$.
   Correspond to 'large' black holes.

$$\mathcal{S}_{\mathrm{macro}} = \pi \sqrt{p^2 q^2 - (pq)^2} + \cdots$$

$\frac{1}{4}$-BPS states in $\mathcal{N}=4$ compactifications

$\frac{1}{4}$-BPS states of Het/T^6: bound states of fundamental strings (electric) and heterotic five-branes (magnetic).

State degeneracy:

$$d(p,q) = \oint d\rho d\sigma dv \frac{e^{i\pi[\rho p^2+\sigma q^2+(2v-1)pq)]}}{\Phi_{10}(\rho,\sigma,v)}$$

Contour integral in the Siegel upper half space:

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix}$$ complex, symmetric, positive definite imaginary part.

$\Phi_{10} =$ weight 10 Siegel cusp form (generalizes η^{24}).

(Strong) motivation for the above formula:

- Dual type-II picture: worldvolume theory of NS5 brane = string field theory (free limit sufficient).
  `R. Dijkgraaf, E. Verlinde and H. Verlinde (1996)`
- Using the D1-D5 system: `S. Shih, A. Strominger and X. Yin (2005)`.

CHL models

Extension to CHL models ($\mathcal{N} = 4$ orbifolds): Φ_{10} is replaced by a cusp form Φ_k of weight k, where $(k+2)(N+1) = 24$ and $N = 1, 2, 3, 5, 7$ is the order of the orbifold twist.

$$d(p,q) = \oint d\rho d\sigma dv \, \frac{e^{i\pi[\rho p^2 + \sigma q^2 + (2v-1)pq]}}{\Phi_k(\rho, \sigma, v)}$$

D.P. Jatkar and A. Sen (2005)

$\mathcal{N}=4$ Supergravity in $\mathcal{N}=2$ formalism

The relevant subsector of the $\mathcal{N}=4$ theory is described by $\mathcal{N}=2$ vector multiplets with prepotential

$$F(Y,\Upsilon) = -\frac{Y^1 Y^a \eta_{ab} Y^b}{Y^0} + F^{(1)}(S)\Upsilon$$

where $S=-i\frac{Y^1}{Y^0}$ is the heterotic dilaton, $a=2,\ldots,n$.
Note $F^{(g>2)}=0$ and $F^{(1)}=F^{(1)}(S)$.

Dilaton S is T-duality invariant and transforms under S-duality as:

$$S \rightarrow \frac{aS-ib}{icS+d}$$

Scalar products $(q^2,p^2,p\cdot q)$ of the electric and magnetic charges $q\in\Gamma$, $p\in\Gamma^*$ are T-duality invariant scalar products and transform in the $\mathbf{3}$-representation of $SL(2,\mathbb{Z})_S$.

Reduced variational principle

All attractor equations can be solved except those which determine the dilaton S. The dilaton attractor equations determine the critical points of the reduced entropy function

$$\Sigma(S,\overline{S},p,q) = -\frac{q^2 - ip\cdot q(S-\overline{S}) + p^2|S|^2}{S+\overline{S}} + 4\Omega(S,\overline{S},\Upsilon,\overline{\Upsilon})$$

(We absorbed $F^{(1)}(S)$ into $\Omega(S,\overline{S},\Upsilon,\overline{\Upsilon})$.)

Dilaton attractor equations:

$$\partial_S\Sigma = 0 = \partial_{\overline{S}}\Sigma$$

Entropy

$$\mathrm{S}_{\mathrm{macro}}(p,q) = \pi\Sigma_*$$

is manifestly T- and S-duality invariant if Ω is S-duality invariant.

`G.L. Cardoso, B.de Wit and T.M. (1999), G.L. Cardoso, B.de Wit, J. Käppeli and T.M. (2006).`

$\mathcal{N} = 4$ BPS black holes

At the two-derivative level, BPS black hole entropy is:

$$\mathrm{S}_{\mathrm{macro}} = \pi \sqrt{q^2 p^2 - (q \cdot p)^2}$$

M. Cvetic and D. Youm, (1995), E. Bergshoeff, R. Kallosh and T. Ortin (1996).

Two cases:

1. Dyonic $\frac{1}{4}$-BPS black holes with non-vanishing area, 'large black holes.'
   $q^2 p^2 - (q \cdot p)^2 \neq 0 \Leftrightarrow M = |Z_1| > |Z_2|$.
2. 'Electric' $\frac{1}{2}$-BPS black holes with vanishing area (at leading order), 'small black holes.'
   $q^2 p^2 - (q \cdot p)^2 = 0 \Leftrightarrow M = |Z_1| = |Z_2|$.

Large black holes

Evaluation of

$$d(p,q) = \oint d\rho d\sigma dv \frac{e^{i\pi[\rho p^2+\sigma q^2+(2v-1)pq)]}}{\Phi_{10}(\rho,\sigma,v)}$$

at leading order (saddle point evaluation without fluctuation determinant) gives

$$\mathrm{S}_{\mathrm{micro}} = \log d(p,q) \approx \pi\sqrt{q^2p^2-(q\cdot p)^2} \approx \mathrm{S}_{\mathrm{macro}}$$

`R. Dijkgraaf, E. Verlinde and H. Verlinde (1996)`

This extends to CHL models.

R^2- and nonholomorphic corrections

R^2 corrections and nonholomorphic corrections are encoded in

$$\Omega = \frac{1}{256\pi}\left[\Upsilon \log \eta^{24}(S) + \overline{\Upsilon} \log \eta^{24}(\overline{S}) + \tfrac{1}{2}(\Upsilon + \overline{\Upsilon}) \log(S + \overline{S})^{12}\right]$$

`J.A. Harvey and G.W. Moore (1996)`

Note:

$$\log \eta^{24}(S) = -2\pi S - 24 e^{-2\pi S} + \mathcal{O}(e^{-4\pi S})$$

Infinite series of space-time instanton corrections.

Saddle point evaluation (including fluctuation determinant) of

$$d(p,q) = \oint d\rho d\sigma dv \, \frac{e^{i\pi[\rho p^2 + \sigma q^2 + (2v-1)pq)]}}{\Phi_{10}(\rho,\sigma,v)}$$

gives $\mathrm{S}_{\mathrm{micro}}(p,q) \approx \pi\Sigma_* \approx \mathrm{S}_{\mathrm{macro}}(p,q)$.

Saddle point equations = dilaton attractor equations.

Semiclassical result, includes the non-perturbative terms in Ω.

`G.L. Cardoso, B. de Wit, J. Käppeli and T.M. (2004)`

Testing (modified) OSV

Evaluate mixed partition function $Z(p,\phi)=\sum_q d(p,q)e^{\pi q_I \phi^I}$ using integral representation of $d(p,q)$. Result:

$$Z(p,\phi)=\sum_{\text{shifts}}\sqrt{\tilde{\Delta}(p,\phi)}e^{\pi\mathcal{F}_{\text{mixed}}(p,\phi)}$$

(sum over shifts to enforce periodicity), with (mixed) free energy

$$\mathcal{F}_{\text{mixed}}(p,\phi)=\tfrac{1}{2}(S+\overline{S})\left(p^a\eta_{ab}p^b-\phi^a\eta_{ab}\phi^b\right)-i(S-\overline{S})p^a\eta_{ab}\phi^b+4\Omega(S,\overline{S},\Upsilon,\overline{\Upsilon})$$

Here

$$\Omega=\frac{1}{256\pi}\left[\Upsilon\log\eta^{24}(S)+\overline{\Upsilon}\log\eta^{24}(\overline{S})+\tfrac{1}{2}(\Upsilon+\overline{\Upsilon})\log(S+\overline{S})^{12}\right]$$

includes both R^2 and nonholomorphic corrections.
Measure factor agrees asymptotically with prediction from modified OSV conjecture:

$$\tilde{\Delta}\approx\Delta^-$$

D. Shih and X. Yin (2005), G.L. Cardoso, B. de Wit, J. Käppeli and T.M. (2006). (Valid for CHL models as well.)

Remarks

- Highly non-trivial test of OSV conjecture, though restricted to the semi-classical limit (no statement about strong version of the conjecture). Presence of a non-trivial measure factor has been established.
- The asymptotic holomorphic factorization

$$Z_{\rm BH} \approx |Z_{\rm top}|^2$$

can be explained for certain black holes in $\mathcal{N} = 2$ compactifications, using AdS^3/CFT_2 and a Rademacher-Jacobi expansion for elliptic genera (=BPS partition functions). It is due to independent contributions from both branes and antibranes ($M2/\overline{M2}$ or $D2/\overline{D2}$).
`D. Gaiotto, A. Strominger and X. Yin, hep-th/0602046, P. Kraus and F. Larsen, hep-th/0607138, C. Beasley, D. Gaiotto, M. Guica, L. Huang, A. Stominger and X. Yin, hep-th/0608021, J. de Boer, M.C.N. Cheng, R. Dijkgraaf, J. Manshot and E. Verlinde, hep-th/0608059`.
- The refined analysis of `F. Denef and G. Moore, hep-th/0702146` has identified a microscopic measure factor, which agrees with ours in the semi-classical limit.

Other higher derivative terms

The Weyl multiplet encodes a specific class of higher derivative terms:

$$R^2 F^{2g-2} + \text{susy transformed} \ .$$

Other higher derivative terms should contribute to the entropy as well.

Evidence for particular importance of R^2-terms: (i) relation to topological string, (ii) success in matching $\mathrm{S}_{\mathrm{macro}} = \mathrm{S}_{\mathrm{micro}}$. (However: resolution of some null singularities in type-II compactifications seems to require R^4-terms.)

Observation: one can substitute the Gauss-Bonnet term for the whole set of supersymmetric R^2-terms.

`K. Behrndt, G.L. Cardoso and S. Mahapatra, NPB 732 (2006) 200, hep-th/0506251, A. Sen, JHEP 03 (2006) 008, hep-th/0508042.`

Entropy appears to be robust!? Universality?

Other higher derivative terms (2)

Progress on explicit construction of further higher derivative terms (using superconformal calculus).

`B. de Wit and F. Saueressig, hep-th/0606148`

Observe cancellations in supersymmetric backgrounds.

4d $\longrightarrow$ 5d lift gives black holes with $AdS^3 \times S^2$ horizon geometry. Matching of $\mathrm{S}_{\mathrm{macro}}$ and $\mathrm{S}_{\mathrm{micro}}$ can be established by (i) using the AdS^3/CFT_2 correspondence and (ii) by matching anomalies R^2-terms cover precisely the relevant contributions.

`P. Kraus and F. Larsen, JHEP 09 (2005) 034, hep-th/0506176`.

Nice lectures on application of AdS_3/CFT_2 correspondence on 4d and 5d black holes: `P. Kraus, hep-th/0609074`.

Critical discussion of strengths and limits of AdS_3/CFT_2 approach: `A. Dabholkar, A. Sen and S.P. Trivedi, hep-th/0611143`.

Non-supersymmetric Black Holes

Attractor mechanism also works for extremal black holes (with $AdS^2 \times S^2$ horizon geometry), which are not supersymmetric. This is again independent of the details of the action, and holds for generally covariant higher derivative theories.
Elegant formalism, based on an entropy function and Wald's entropy formula:
`A. Sen, hep-th/0506177, hep-th/0508042, ...`

Relation between Sen's formalism and the variational principle reviewed here has been explored by G.L. Cardoso, B. de Wit and S. Mahapatra.

Why is $AdS^2 \times S^2$ more essential than supersymmetry? Plausibility argument: $AdS^2 \times S^2$ is a flux compactification on S^2. Flux generates scalar potential which fixes scalars. Applies to non-superysmmetric theories, but also supersymmetric theories can have non-supersymmetric vacua.

Examples for non-supersymmetric, extremal black holes in supersymmetric compactifications:
`K. Goldstein, N. Iizuka, R.P. Jena and S.P. Trivedi, Phys. Rev. D 72 (2005) 124021, hep-th/0507096, R. Kallosh, JHEP 12 (2005) 022, hep-th/0510024, ...`

Dimensional reduction of (stationary) black holes

(BPS and non-BPS) space-times can be dimensionally reduced using that they have (commuting) Killing vectors.

- Static case: reduction to transverse radial coordinate $\rightarrow$ quantum mechanics with black hole potential. `G. Gibbons, S. Ferrara and R. Kallosh (1997)`.
- Spatial reduction, e.g. relating 4d and 5d black holes `D. Gaiotto, A. Strominger and X. Yin, hep-th/0503217`.
- Timelike Killing vector $\rightarrow$ dimensional reduction from $D+1$ to D (Euclidean) dimensions. I.p. black holes $\rightarrow$ instantons.
  Well known in GR, `G. Neugebauer and D. Kramer (1969)`, . . . also used in supergravity/string theory, `G. Clement and D.V. Gal'tsov (1996)`, ...
  Time-like reduction used recently (in OSV context) by `M. Günaydin, A. Neitzke, B. Pioline and A. Waldron, arXiv:0707.0267`.
- T-duality combines dimensional reduction and lifting. E.g., 4d black holes $\leftrightarrow$ 4d instantons, `K. Behrndt, I. Gaida, D. Lüst, S. Mahapatra and T.M. (1997)`.
  (Hull's $*$-type string theories: reduce over time and lift again to time-like direction.)

Timelike reductions, Euclidean special geometry

- Well known: timelike dimensional reduction leads to scalar manifolds with indefinite signature.
- This leads to modifications of special geometry, and of the r-map and c-map (which relate supermultiplets in 5d/4d/3d).
  A systematic study of 'Euclidean special geometry' was initiated in `V. Cortés, C. Mayer, T.M. and F. Saueressig, JHEP 0403:028 (rigid vector multiplets) and JHEP 0506:025 (rigid hypermultiplets).`
- Here we discuss the dimensional reduction of 5d supergravity with vector multiples to four (Euclidean) dimensions, leading to (projective) special para-Kähler geometry. This gives a 'temporal version' of the r-map.
  `V. Cortés and T.M. in preparation.`

5d supergravity

Bosonic Lagrangian

$$\hat{\mathbf{e}}^{-1}\hat{\mathcal{L}} \quad = \quad \frac{1}{2}\hat{R} - \frac{3}{4}a_{ij}\partial_{\hat{\mu}}h^i\partial^{\hat{\mu}}h^j - \frac{1}{4}a_{ij}\mathcal{F}^i_{\hat{\mu}\hat{\nu}}\mathcal{F}^{j\hat{\mu}\hat{\nu}} + \frac{\hat{\mathbf{e}}^{-1}}{6\sqrt{6}}c_{ijk}\epsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\lambda}}\mathcal{F}^i_{\hat{\mu}\hat{\nu}}\mathcal{F}^j_{\hat{\rho}\hat{\sigma}}\mathcal{A}^k_{\hat{\lambda}} \ .$$

where $i, j, = 1, \ldots, n_V^{(5)} + 1$ and $\hat{\mu}, \hat{\nu}, \ldots = 0, \ldots 4$.
Scalar manifold $M^{(5)}$ is a real cubic hypersurface $\mathcal{V} = c_{ijk}h^ih^jh^k = 1$ (very special real manifold).

Metric = (pullback of)

$$a_{ij} \ = \ -(\frac{1}{3}\partial_{h^i}\partial_{h^j}\ln\mathcal{V})_{|\mathcal{V}=1} \ .$$

`M. Günaydin, G. Sierra and P. Townsend (1984).`

Perform reduction over space-like direction ($\epsilon = -1$) and time-like direction ($\epsilon = +1$) in parallel.

Dimensional reduction of 5d supergravity

Reduced Lagrangian (in suitable variables):

$$
\begin{aligned}
\mathbf{e}^{-1}\mathcal{L}^{\epsilon} \quad = \quad & \frac{1}{2}R - g_{ij}\left(\partial_\mu x^i \partial^\mu x^j - \epsilon \partial_\mu y^i \partial^\mu y^j\right) \\
& + \epsilon \left(\frac{1}{4} cyyy \left(\frac{1}{6} + \frac{2}{3}\, gxx \right) F^0 \cdot F^0 - \frac{1}{3}\, cyyy\, (gx)_i F^0 \cdot F^i + \frac{1}{6}\, cyyy\, g_{ij}\, F^i \cdot F^j \right) \\
& - \frac{\mathbf{e}^{-1}}{12} \left(cxxx\, F^0 \cdot \tilde{F}^0 - 3(cxx)_i\, F^i \cdot \tilde{F}^0 + 3(cx)_{ij}\, F^i \cdot \tilde{F}^j \right) \; .
\end{aligned}
$$

where $i, j, \ldots = 1, \ldots, n_V^{(4)} = n_V^{(5)} + 1$ and $cxxx = c_{ijk} x^i x^j x^k$, etc.
Lorentz indices: suppressed.
Scalar metric $g_{ij} \oplus (-\epsilon) g_{ij}$,

$$
g_{ij} = \epsilon\, \frac{3}{2} \left(\frac{(cy)_{ij}}{cyyy} - \frac{3}{2}\, \frac{(cyy)_i (cyy)_j}{(cyyy)^2} \right)
$$

does not depend on the ‘axions’ x^i, which come from the 5d gauge fields.

For time-like reduction the signature of the scalar manifold is split: $(+)^{n_V^{(4)}}(-)^{n_V^{(4)}}$. ($\epsilon = -1$ for space-like and $\epsilon = 1$ for time-like reduction.)

ϵ-complex structures

Structure of the Lagrangian suggests to define:

$$z^i = x^i + i_\epsilon y^i \; .$$

where

$$i_\epsilon^2 = \epsilon = \left\{ \begin{array}{ll} -1 & (\text{complex structure}) \\ +1 & (\text{para-complex structure}) \end{array} \right\} \epsilon - \text{complex structure}$$

Definition: An almost ϵ-complex structure on an smooth manifold M is an endomorphism field $I \in \Gamma(\text{End}\, TM)$, such that (i) $I^2 = \epsilon$ and (ii) the eigendistributions $\text{ker}(\text{Id} \mp I)$ of I have the same rank (implying that dimension of M must be even).

Similarly: (integrable) ϵ-complex, ϵ-hermitean, ϵ-Kähler and, in fact, (affine/projective) ϵ-Kähler. See `V. Cortés, C. Mayer, T.M. and F. Saueressig, JHEP 0403:028, JHEP 0506:025 and V. Cortés and T.M. in prepration`.

4d Lagrangian

4d Lagrangian can be brought to the following standard form

$$\mathbf{e}^{-1}\mathcal{L}^{(4)} = \frac{1}{2}R - G_{i\bar{j}}\partial_\mu z^i \partial^\mu \overline{z}^{\bar{j}} + \frac{1}{4}\text{Im}\mathcal{N}_{IJ}F^I F^J + \frac{\mathbf{e}^{-1}}{4}\text{Re}\mathcal{N}_{IJ}F^I \tilde{F}^J \ ,$$

where $z^i = x^i + i_\epsilon y^i$ are ϵ-complex scalar fields. $G_{i\bar{j}}$ and $\mathcal{N}_{IJ}$ are given by standard formulae in terms of a (cubic) ϵ-holomorphic prepotential

$$F(X) = -\frac{1}{6}c_{ijk}\frac{X^i X^j X^k}{X^0} \ ,$$

where $X^I = (X^0, X^i)$ are valued in an ϵ-complex cone over the scalar manifold $M^{(4)\epsilon}$. Relation to physical scalars:

$$z^i = \frac{X^i}{X^0} \ .$$

Generalized r-map relates very special real to (very) special ϵ-Kähler manifolds:

$$r_\epsilon : M_n^{(5)} \longrightarrow M_{2n+2}^{(4)\epsilon}$$

$M_{2n+2}^{(4)\epsilon=\pm 1}$ are two real sections of the same complex-Riemannian space.

Outlook: Instantons

- Instantons (finite action solutions to the Euclidean equations of motion) correspond to harmonic maps onto completely geodesic, completely isotropic submanifolds of $M^{(4)\epsilon=+1}$. `P. Breitenlohner, D. Maison and G. Gibbons (1988).`
- We have contructed a class of instanton solutions, and studied their dual description, where the axions x^i are replaced by tensor fields (vector-tensor multiplets).
- These can then be uplifted to 5d stationary solutions.
- We believe that some of our solutions are invariant under Euclidean supersymmetry, but there should exist non-supersymmetric solutions as well.
- Conceptual issues: timelike reduction and Wick rotation give different Euclidean actions (which are real sections of the same complex action). Both seem to play a role, moreover there is a dual description in terms of a scalar/tensor action.
- Relation to black hole partition functions (OSV), minisuperspace approximations, hidden symmetries, . . .

Counting $\frac{1}{2}$-BPS states

Partition function for $\frac{1}{2}$-BPS states:

$$d(q) = d(q^2) = 16 \oint d\tau \, \frac{\exp(i\pi\tau q^2)}{\eta^{24}(\tau)}$$

where $\tau = \tau_1 + i\tau_2 \in \mathcal{H}$ (upper half plane) and $\eta(\tau) =$ Dedekind η-function.
η^{24} is a modular form of weight 12.
Variant of the partitioning problem of `G.H. Hardy and S. Ramanujan (1918)`.

Evaluation through Rademacher expansion, aka. Farey tail expansion.
`H. Rademacher (1938), see R. Dijkgraaf, J. Maldacena, G. Moore and E. Verlinde (2000).`

$$d(q^2) = 16 \sum_{c=1}^{\infty} c^{-14} \mathrm{Kl}(\tfrac{1}{2}|q^2|, -1; c) \hat{I}_{13} \left(\frac{4\pi}{c} \sqrt{\frac{1}{2}|q^2|} \right)$$

$\hat{I}_{13} =$ modified Bessel functions, Kl= 'Kloosterman sums'. Contributions $c > 1$ are exponentially suppressed for large $|q^2|$.

Counting $\frac{1}{2}$-BPS states (2)

Leading term

$$d(q^2) = 16\ \hat{I}_{13}(4\pi\sqrt{\tfrac{1}{2}|q^2|})$$

can be further expanded:

$$\mathrm{S}_{\mathrm{micro}}(q^2) = \log d(q^2) \approx 4\pi\sqrt{\tfrac{1}{2}|q^2|} - \frac{27}{4}\log|q^2| + \frac{15}{2}\log(2) - \frac{675}{32\pi|q^2|} + \cdots$$

(Taken from `A. Dabholkar, F. Denef, G. Moore and B. Pioline (2005)`)

First two terms = Cardy formula = saddle point evaluation of the integral representation.
(First term = value of integrand at saddle point, second term = fluctuation determinant.)

OSV for small black holes

Solution based on two-derivative effective action:

$$\mathrm{S}_{\mathrm{macro}} = \pi\sqrt{p^2 q^2 - (p\cdot q)^2} = 0 \quad \text{for} \quad p = 0$$

Area: $A = 0$, null singularity.
Scalars attracted to the boundary of moduli space, i.p. dilaton $S = \infty$.

Entropy disagrees with leading order string state counting:

$$\mathrm{S}_{\mathrm{micro}} \approx 4\pi\sqrt{\tfrac{1}{2}|q^2|}$$

OSV for small black holes (2)

First subleading correction is the (heterotic) tree level R^2-term encoded in

$$\log \eta^{24}(S) = -2\pi S + \mathcal{O}(e^{-2\pi S})$$

Stringy cloaking of the null singularity:

$$A = 8\pi \sqrt{\tfrac{1}{2}|q^2|} \neq 0$$

R^2-corrections generate a finite horizon.

Entropy

$$\mathrm{S}_{\mathrm{macro}} = \frac{A}{4} + \text{Wald's correction} = \frac{A}{4} + \frac{A}{4} = \frac{A}{2} = 4\pi \sqrt{\tfrac{1}{2}|q^2|}$$

agrees with leading order $\mathrm{S}_{\mathrm{micro}}$. Wald's modification of the area law is crucial.
`A. Dabholkar, R. Kallosh and A. Maloney (2004).`

What about subleading terms in the entropy?

OSV for small black holes (3)

Including the non-holomorphic corrections:

$$\begin{aligned} \mathrm{S}^{(\mathrm{Wald})}_{\mathrm{macro}} &= 4\pi\sqrt{\tfrac{1}{2}|q^2|} - 6\log|q^2| + \cdots \\ \mathrm{S}^{(\mathrm{Cardy})}_{\mathrm{micro}} &= 4\pi\sqrt{\tfrac{1}{2}|q^2|} - \frac{27}{4}\log|q^2| + \cdots \end{aligned}$$

However, both entropies refer to different ensembles, according to OSV. Our modified version of the conjecture implies

$$\mathrm{S}_{\mathrm{micro}} = \mathrm{S}_{\mathrm{macro}} + \log\sqrt{\frac{\Delta^-}{\Delta^+}}$$

But for electric black holes

$$\begin{aligned} \Delta^- &= 0 \quad \text{up to non-holomorphic terms and instantons} \\ \Delta^+ &= 0 \quad \text{up to instantons} \end{aligned}$$

OSV for small black holes (4)

Cannot perform saddle point approximation of the full integral, because leading order solution is singular.

We can still test the idea that OSV has to be modified by our measure factor Δ^- and by nonholomorphic terms by evaluating

$$\exp(\mathrm{S}_{\mathrm{micro}}) = d(p^1, q) \approx \int d\phi \sqrt{\Delta^-(p^1,\phi)} e^{\pi[\mathcal{F}_E(p^1,\phi) - q_I \phi^I]}$$

when including the nonholomorphic terms in Δ^-.
Note: p^1 is an electric charge (for the heterotic string).

Neglecting instanton corrections, we find:

$$d(p^1, q) \approx \int \frac{dS d\overline{S}}{(S+\overline{S})^{k+4}} \sqrt{S + \overline{S} - \frac{k+2}{2\pi}} \exp\left[-\frac{\pi q^2}{S+\overline{S}} + 2\pi(S+\overline{S}) \right]$$

where $k = 10$ for Het/T^6 and other values for CHL models.

OSV for small black holes (5)

Thus (for $k = 10$)

$$\begin{aligned} \mathrm{S}^{(\mathrm{mod.OSV})}_{\mathrm{predicted}} &\approx \hat{I}_{13-\frac{1}{2}}(4\pi\sqrt{\tfrac{1}{2}|q^2|}) \approx 4\pi\sqrt{\tfrac{1}{2}|q^2|} - \frac{13}{2}\log|q^2| + \cdots \\ \mathrm{S}_{\mathrm{micro}} &\approx \hat{I}_{13}(4\pi\sqrt{\tfrac{1}{2}|q^2|}) \approx 4\pi\sqrt{\tfrac{1}{2}|q^2|} - \frac{27}{4}\log|q^2| + \cdots \end{aligned}$$

G.L. Cardoso, B. de Wit, J. Käppeli and T.M., hep-th/0601108.
Slight but systematic mismatch of subleading $\log$ and inverse power corrections. Same for CHL models. Shift is due to the measure. Maybe there is no measure?

Unmodified OSV conjecture: no measure, no nonholomorphic terms:

$$d(p^1, q) \approx \int d\phi e^{\pi[\mathcal{F}_{OSV}(p^1,\phi) - q_I\phi^I]} \approx (p^1)^2 \hat{I}_{13}(4\pi\sqrt{\tfrac{1}{2}|q|})$$

Factor $(p^1)^2$ spoils T-duality. (Some) Measure needed.
Index $\nu = 13$ of Bessel function requires to truncate (unmodified) OSV integral to 24 potentials. The dyonic case works with 28.

A. Dabholkar, F. Denef, G. Moore and B. Pioline (2005)

OSV for small black holes (6)

- For small black holes, there are discrepancies at semiclassical limit beyond the leading term.
- A measure factor seems to be needed for duality invariance.
- Problems are related to singular behaviour of 'leading order' solution. How to set up a well defined expansion?
- When including instanton corrections, the structure of OSV-type integrals looks different from integral representations of state degeneracies. Unclear how strong version of the conjecture could work.

OSV for small black holes (7)

One more result from the comprehensive study of small $\mathcal{N} = 4$ and $\mathcal{N} = 2$ black holes performed by `A. Dabholkar, F. Denef, G. Moore and B. Pioline (2005)`. (See also nice review by `B. Pioline, hep-th/0607227`)

- In twisted sectors of $\mathcal{N} = 2$ orbifolds, absolute and indexed degeneracies are equal and agree with OSV.
- In the untwisted sector of $\mathcal{N} = 2$ orbifolds, the leading order absolute degeneracies agree with OSV. Indexed degeneracies are exponentially smaller.
- Proposal: absolute degeneracy = 'appropriate' index. (Idea: at finite coupling, many BPS states decay on lines of marginal stability.) `F. Denef and G. Moore, hep-th/0702146`.