A. Mironov, T. Tomaras

n-particle amplitudes in planar $\mathcal{N}=4$ SYM theory

planar : $SU(N)$

$N = \infty$

$\lambda = g^2 N$ finite

$\mathcal{N}=4$ SYM:

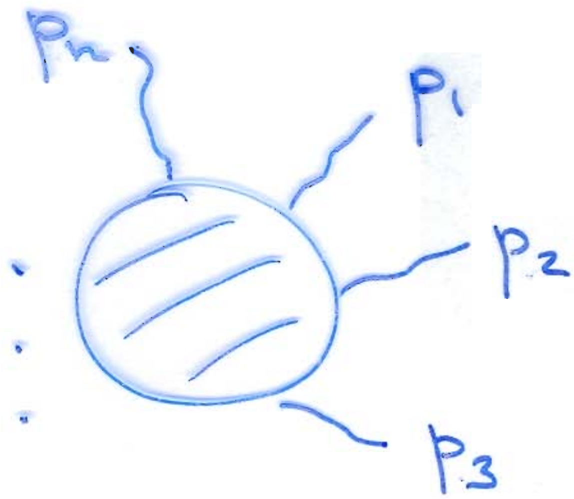
UV-finite

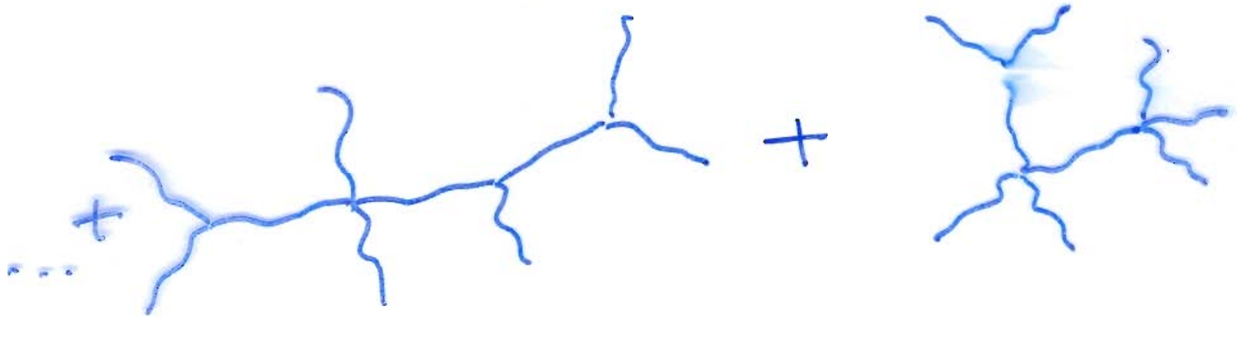
Exactly solvable?

Exactly "solved"!

$$\mathcal{A}(p_1,\dots,p_n) =$$

$$= \mathcal{A}_{tree}\,\underbrace{\mathcal{A}_{IR}\,\mathcal{A}_{finite}}_{\text{Lorentz scalars}}$$





$$\lambda = g^2 N$$

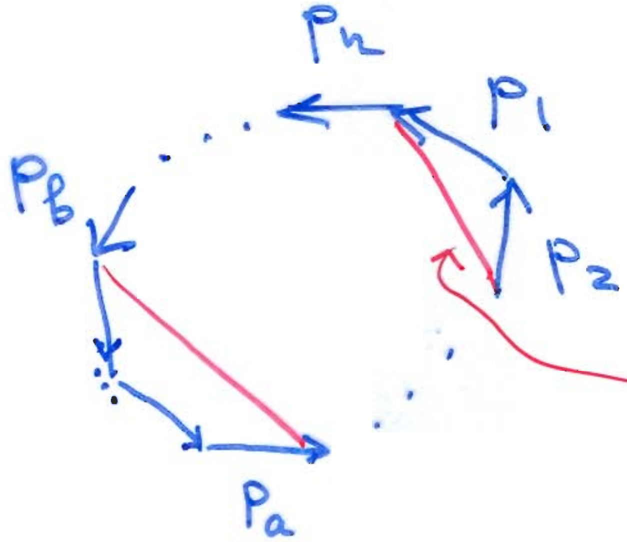
$$\epsilon$$

$$\log \mathcal{A}_{IR} \sim$$

$$\sim \sum_a \frac{1}{\epsilon^2}\left(\frac{p_a p_{a+1}}{\mu^2}\right)^{-\epsilon}$$

BDS conjecture [Bern, Dixon, Smirnov]

$$\mathcal{A}_{finite} = \exp\left(\gamma(\lambda)\, F_n^{(1)}(p_1 \ldots p_n)\right)$$



$(p_1+p_2)^2 = s$

$(p_2+p_3)^2 = t$

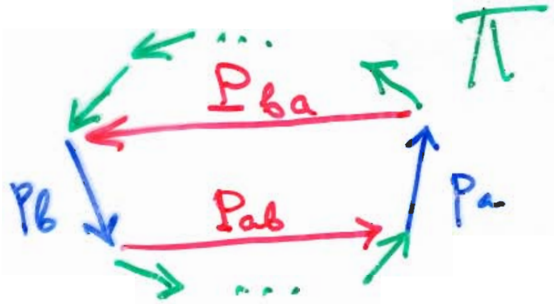
$$t_{ab} = (p_a + \ldots + p_{b-1})^2$$
$$\|$$
$$t_a^{[b-a]}$$

$$t_a^{[r]}$$

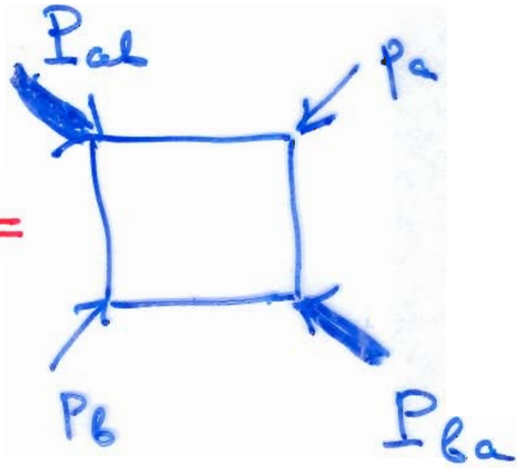
$$\tau_a^{[r]} = \log t_a^{[r]}$$

BDS formula. I.

$$F_n^{(1)}(\underbrace{p_1,\dots,p_n}_{\cdots p_a \cdots p_b \cdots}) = \sum_{a<b} F^{2me}(p_a, P_{ab}, p_b, P_{ba})$$



$F^{2me} =$



$$p_a^2 = p_b^2 = 0$$

$$P_{ab}^2 \neq 0, \quad P_{ba}^2 \neq 0$$

$$\sim \iiiint_0^1 \frac{d\beta_1 \dots d\beta_4 \;\delta(1-\beta_1-\beta_2-\beta_3-\beta_4)}{\Big(\underset{\underset{(p_a+P_{ab})^2}{\uparrow}}{s}\beta_1\beta_3 + \underset{\underset{P_{ab}^2}{\uparrow}}{P^2}\beta_3\beta_4 + \underset{\underset{(P_{ab}+p_b)^2}{\uparrow}}{t}\beta_2\beta_4 + \underset{\underset{P_{ba}^2 \,=\, (p_a+P_{ab}+p_b)^2}{\uparrow}}{Q^2}\beta_1\beta_2\Big)^{2+\epsilon}}$$

$$F^{2me} = \frac{2i}{4\pi^2}\,\frac{\Gamma(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)}\,\frac{1}{st-P^2Q^2}\,\times$$

$$\times\left\{\frac{1}{\epsilon^2}\left[\left(\frac{4\pi\mu^2}{s}\right)^{\epsilon}+\left(\frac{4\pi\mu^2}{t}\right)^{\epsilon}\pm\left(\frac{4\pi\mu^2}{P^2}\right)^{\epsilon}-\left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon}\right]\right.$$

$$+\,Li_2\left(1-\frac{P^2Q^2}{st}\right)-Li_2\left(1-\frac{P^2}{s}\right)-Li_2\left(1-\frac{P^2}{t}\right)$$

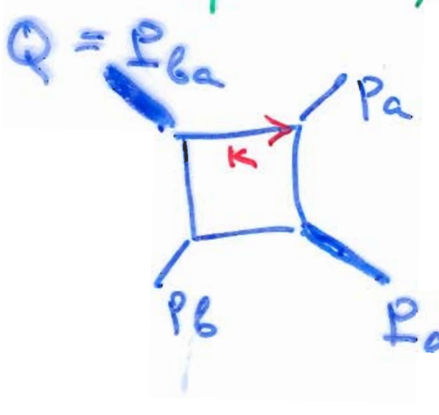
$$\left.-\,Li_2\left(1-\frac{Q^2}{s}\right)-Li_2\left(1-\frac{Q^2}{t}\right)\right\}\ \text{— BDK}$$

$$+\,Li_2(1-as)+Li_2(1-at)-Li_2(1-aP^2)-$$

$$\left.-\,Li_2(1-aQ^2)\right\}$$

$$a = \frac{s+t-P^2-Q^2}{st-P^2Q^2}$$

C. Duplancic, B. Nizic



$$= \int \frac{d^{4+\epsilon}k}{k^2(k+p_a)^2(k+p_a+P_{ab})^2(k-P_{ba})^2}$$

$$\mathrm{Li}_{\underset{\overset{\uparrow}{p}}{2}}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^{\underset{\overset{\uparrow}{p}}{2}}} = -\int^{z} \log(1-z)\,\frac{dz}{z}$$

$$\mathrm{Li}_2(1) = \sum \frac{1}{k^2} = \zeta(2) = \frac{\pi^2}{6}$$

$$\sin \pi x = \pi x \prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2}\right)$$

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = -\log(1-z)\log(z) - \frac{\pi^2}{6}$$

$$\mathrm{Li}_2(z) + \mathrm{Li}_2\left(\frac{1}{z}\right) = -\frac{1}{2}\left(\log(-z)\right)^2 - \frac{\pi^2}{6}$$

BDS formula. II.

$$\mathcal{A}(p_1 \dots p_n) = \mathcal{A}_{tree}\, \mathcal{A}_{IR}\, e^{\gamma(\lambda) F_n^{(1)}(p_1 \dots p_n)}$$

$$F_n^{(1)}(p_1 \dots p_n) = \sum_{a<b} F^{2me}(p_a, P_{a\ell}, p_b, P_{b\ell})$$

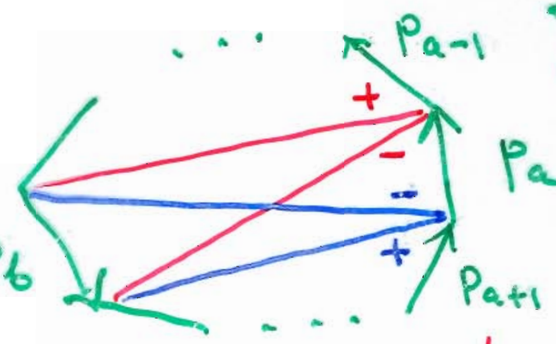
$$= \sum_{a=1}^{n} \Bigg(-\frac{1}{4} \sum_{r=2}^{n-4} Li_2\left(1 - \frac{t_a^{[r]}\, t_{a+1}^{[r+2]}}{t_a^{[r+1]}\, t_{a+1}^{[r+1]}}\right)$$

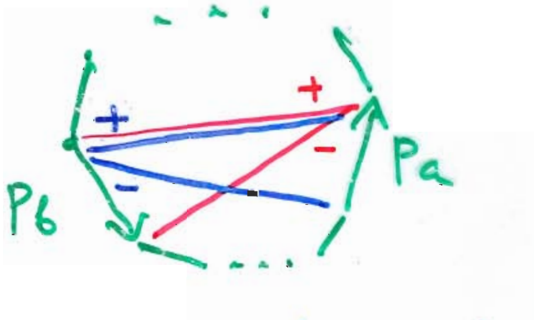
$$- \frac{1}{2} \sum_{r=2}^{[n/2]-1} \log \frac{t_a^{[r+1]}}{t_a^{[r]}} \log \frac{t_a^{[r+1]}}{t_{a+1}^{[r]}}$$

$$- \frac{1}{8} \cancel{\Sigma} \log \frac{t_a^{[n/2]}}{t_{a+\frac{n}{2}+1}^{[n/2]}} \log \frac{t_{a+1}^{[n/2]}}{t_{a+\frac{n}{2}}^{[n/2]}} \Bigg)$$

even n

II





$$Li_2\left(1 - \frac{t_{long}\, t_{short}}{t_{medium_1}\, t_{medium_2}}\right) \qquad \log \frac{t_{long}}{t_{m_1}} \log \frac{t_{long}}{t_{m_2}}$$

BDS formula. III

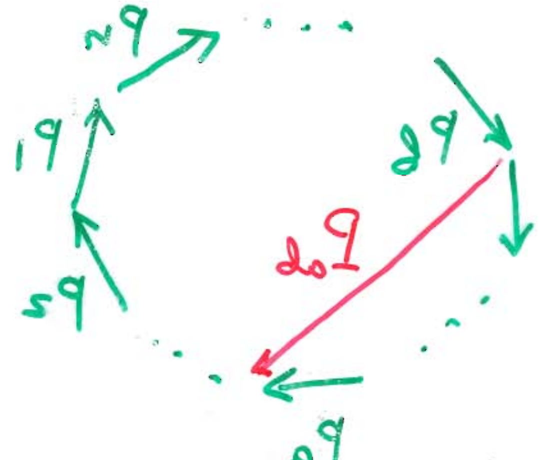
A. Brandhuber, P. Heslop, G. Travaglini

$$\mathcal{A} = \mathcal{A}_{tree}\;\mathcal{A}_{IR}\;\mathcal{A}_{finite}$$

$$\mathcal{A}_{IR} = \exp \gamma(\lambda)\, F_n^{(1)}(p_1 \ldots p_n)$$

$$F_n^{(1)} \cong \oint_{\Pi}\oint_{\Pi} \frac{dy^\mu\, dy'_\mu}{(y-y')^{2+\epsilon}}$$

Π



QFT =

= pure geometry

$$= \sum_{a<b} \int_0^1\!\!\int_0^1 \frac{(p_a p_b)\, d\tau_a\, d\tau_b}{[(1-\tau_a)p_a + \tau_b p_b + P_{ab}]^{2+\epsilon}}$$

AdS/CFT = string/gauge duality

QFT: $\mathcal{A} = \mathcal{A}_{tree}\, \mathcal{A}_{IR}\, e^{\gamma(\lambda) F_n^{(1)}}$

$$\gamma(\lambda) = \gamma_1 \lambda + \gamma_2 \lambda^2 + \dots$$

$$F_n^{(1)} = \oint_{\Pi}\oint \frac{dy\, dy'}{(y-y')^{2+\epsilon}}$$

string:

$$\gamma(\lambda) = \tilde{\gamma}_1 \sqrt{\lambda} + \tilde{\gamma}_0 + \tilde{\gamma}_{-1} \frac{1}{\sqrt{\lambda}} + \dots$$

$$F_n^{(1)} = (\text{minimal Area})_\epsilon$$

in AdS_5

$$\frac{-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 + dr^2}{r^2}$$

F. Alday, J. Maldacena:

Minimal area$_\epsilon$ for $n=4$

coincides with $F_4^{(1)} \sim \left(\log \frac{s}{t}\right)^2$

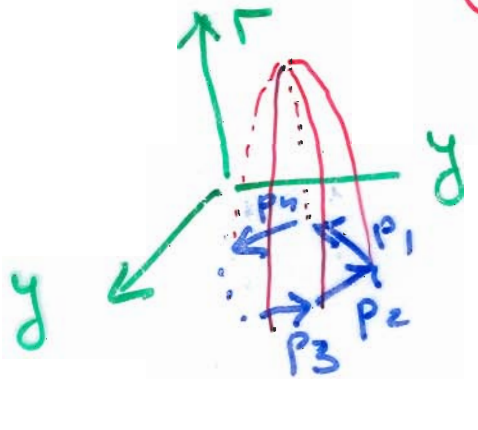
- σ-model action instead of NG

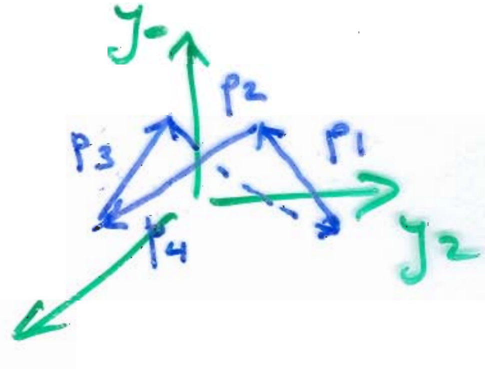
T-duality transform $\partial_i x^\mu \to \epsilon_{ij} \partial_j y^\mu$

$$\frac{dy^2 + dr^2}{r^2}$$

boundary conditions:

at $r=0$ $y \in \Pi$!





- "dimensional" regularization

$$\frac{dy^2 + dr^2}{r^{2+\epsilon}}$$

- KLOV interpolation for $\gamma(\lambda)$

Equations of motion for $SO(4,2)$ σ-model

$$\overset{\shortparallel}{AdS_5}$$

$$Y_+ Y_- + Y^2 = Y_{-1}^2 + \underbrace{Y_0^2 - Y_1^2 - Y_2^2 - Y_3^2}_{Y^2} - Y_4^2 = R^2$$

$$\int \underbrace{\frac{(\partial y)^2 + (\partial r)^2}{r^2}}_{L} d^2u$$

$$\begin{cases} -\partial \frac{1}{r^2} \partial r = \frac{L}{r} \xrightarrow{z = 1/r} \partial^2 z = L z \\ \partial \frac{1}{r^2} \partial y = 0 \end{cases}$$

if $L = \text{const}$ then

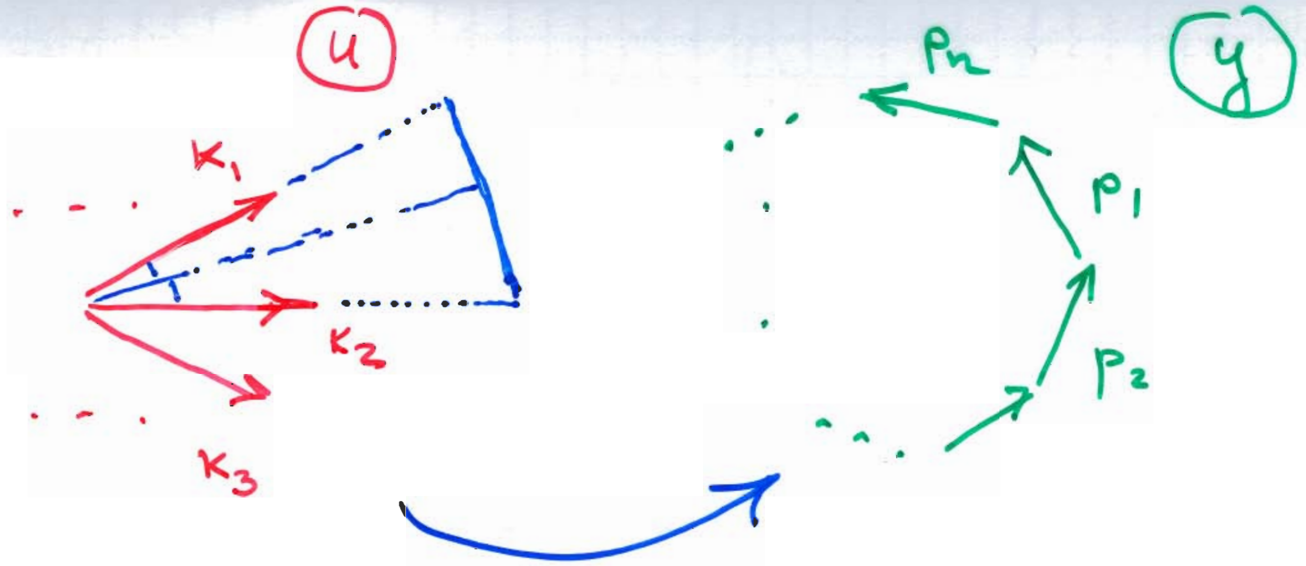
$$z = \sum_a z_a e^{\vec{k}_a \vec{u}}$$

$$k_a^2 = L$$

$$v = \sum_a v_a e^{\vec{k}_a \vec{u}}$$

$$y = r v = \frac{v}{z}$$

$$\partial^2 v = L v$$



$$z \sim z_1 e^{\vec{k}_1 \vec{u}} + z_2 e^{\vec{k}_2 \vec{u}} =$$

$$= e^{(\bar{k}_1 + \bar{k}_2)\vec{u}/2} \left(z_1 t + z_2 \frac{1}{t} \right)$$

$$t = e^{(\bar{k}_1 - \bar{k}_2)\bar{u}/2}$$

$$v \sim e^{(\bar{k}_1 + \bar{k}_2)\bar{u}/2} \left(v_1 t + v_2 \frac{1}{t} \right)$$

$$y^\mu = \frac{v}{z}^\mu = \frac{v_1^\mu t + v_2^\mu \frac{1}{t}}{z_1 t + z_2 \frac{1}{t}}$$

$$t \in (-\infty, +\infty) \longrightarrow$$ segment of a straight line

$$c^\mu y^\nu - c^\nu y^\mu = c^{\mu\nu}$$

// vector p_1^μ

$$p_1 = \frac{v_2}{z_2} - \frac{v_1}{z_1}$$

$$p_a = \frac{v_{a+1}}{z_{a+1}} - \frac{v_a}{z_a}$$

Non-trivial equation: $L = \mathrm{const}$

$$L = \frac{(\partial y)^2 + (\partial r)^2}{r^2}$$

$$z = \frac{1}{r}\ ; \quad y = \frac{v}{z}$$

$$(\partial z)^2 - L z^2 = -(z\,\partial v - v\,\partial z)^2$$

$$z = \sum_{a=1}^{n} z_a e^{\bar{k}_a \bar{u}} \qquad v = \sum_{a=1}^{n} v_a \underbrace{e^{\bar{k}_a \bar{u}}}_{E_a}$$

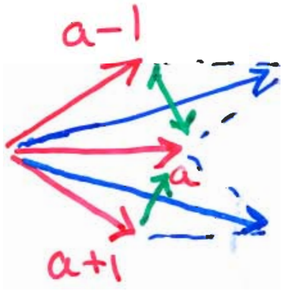
$$\sum_{a,b} z_a z_b \left(L - \vec{k}_a \vec{k}_b\right) E_{a+b} =$$

$$= \sum_{\substack{a<b \\ c<d}} (\vec{K}_{ab} \vec{K}_{cd})\left(\mathcal{P}^{\mu}_{ab}\, \mathcal{P}^{\mu}_{cd}\right) E_{a+b+c+d}$$

$$z\,\partial v - v\,\partial z = \sum_{a,b} z_a v_b\, E_{a+b} \left(\vec{k}_b - \vec{k}_a\right) =$$

$$= -\sum_{a<b} \vec{K}_{ab}\, \mathcal{P}_{ab}\, E_{a+b}$$

$$\mathcal{P}_{ab}: \quad z_a v_b - z_b v_a \overset{!}{=} z_a z_b\, \mathcal{P}_{ab}$$



$$\vec{K}_{a,a+1} \perp \vec{K}_{a,a-1} \qquad \mathcal{P}^2_{a,a+1} \sim p_a^2 = 0$$

$$\not\Rightarrow \qquad \Downarrow$$

$$E_a E_{a+1}\, E_a E_{a-1} \qquad n = 4 \qquad (E_a E_{a+1})^2$$

Regularized action (area_ϵ)

$$\int \underset{\text{const}}{\underset{\shortparallel}{L}}\, d^2u = \infty$$

use solution at $\epsilon=0$ with $r \to r\sqrt{1+\epsilon/2}$

$$S_\epsilon = \int \frac{(\partial r)^2 + (\partial y)^2}{r^{2+\epsilon}}\, d^2u \Longrightarrow$$

$$\Rightarrow \frac{1}{(1+\epsilon/2)^{1+\epsilon/2}} \int \Big(\underset{\text{const}}{\underset{\shortparallel}{L}} + \frac{\epsilon}{2}\frac{(\partial z)^2}{z^2} \Big) z^\epsilon d^2u$$

– does not depend on $V_a \Rightarrow$ on p_a

depends only through eqm

$$\sum_{a<b} z_a z_b t_{ab} = 1$$

$$z_1 z_3 \underset{(p_1+p_2)^2}{\underset{\shortparallel}{s}} + z_2 z_4 \underset{(p_2+p_3)^2}{\underset{\shortparallel}{t}} = 1$$

$$S_\epsilon = \underset{\frac{8}{\epsilon^2 |\sin\phi|}\left(1+\epsilon^2\left(\frac{1}{4}-\frac{\pi^2}{12}\right)\right)}{\underset{\shortparallel}{K_\epsilon}} \Big\{ 1 + \frac{\epsilon}{4}\log(z_1z_2z_3z_4) + \frac{\epsilon^2}{8}\log(z_1z_3)\log(z_2z_4) \Big\}$$

Minimum of S_ϵ in the moduli space

$$\min \left(\frac{1}{\epsilon} \sum_a \log z_a \right) \Bigg|_{\sum_{a<b} z_a z_b t_{ab} = 1}$$

$\Rightarrow$ height function

$$h_4 = \frac{1}{8} \log z_1 z_3 \, \log z_2 z_4$$

$$z_1 = z_3 = \frac{1}{\sqrt{2s}} \left(1 - \frac{\epsilon}{8} \log \frac{s}{t} + O(\epsilon^2) \right)$$

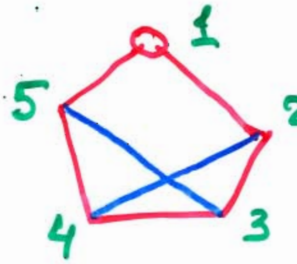
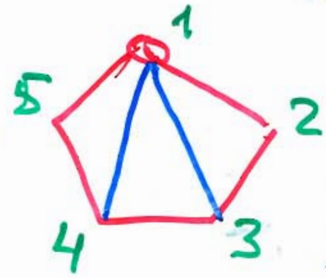
$$z_2 = z_4 = \frac{1}{\sqrt{2t}} \left(1 + \frac{\epsilon}{8} \log \frac{s}{t} + O(\epsilon^2) \right)$$

$$h_4 = \frac{1}{8} \log s \, \log t \rightsquigarrow \frac{1}{4} \left(\log \frac{s}{t} \right)^2$$

$$\frac{\sqrt{\lambda_D c_D}}{2\pi} S_\epsilon = 2^{1+2\epsilon} \frac{\tilde{K}_\epsilon}{\pi \epsilon^2} \Bigg\{ \underbrace{\sqrt{\frac{\lambda \mu^{2\epsilon}}{s^\epsilon}} + \sqrt{\frac{\lambda \mu^{2\epsilon}}{t^\epsilon}}}_{\log A_{IR}} \underbrace{- \frac{\sqrt{\lambda} \epsilon^2}{8} \left(\log \frac{s}{t} \right)^2}_{\sqrt{\lambda} F_4^{(1)}} \Bigg\}$$

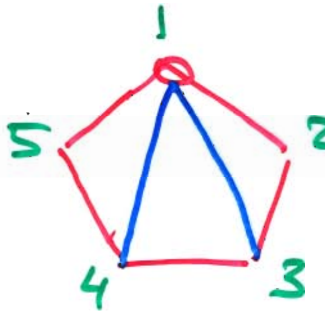
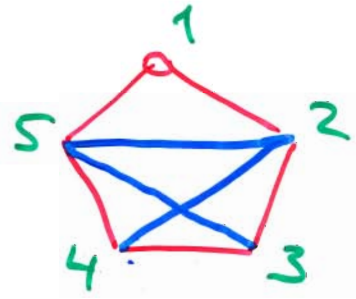
$$A = A_{free} \, A_{IR} \, \exp \left[\gamma(\lambda) F_4^{(1)} \right]$$

Height function for $n=5$

$\mathrm{BDKS} = \sum_{\circ}$  $\cdot$ 

$$= \log\frac{t_{24}}{t_{35}} \cdot \log\frac{t_{14}}{t_{13}} + \text{perms}$$

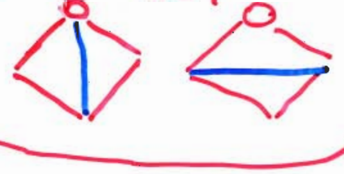
$\not\equiv$ at the minimum of $\sum_a \log z_a \Big|_{\sum_{a<b} t_{ab} z_a z_b = 1}$

$\log z_1 =$  $-$ 

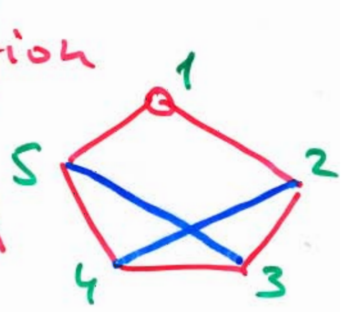
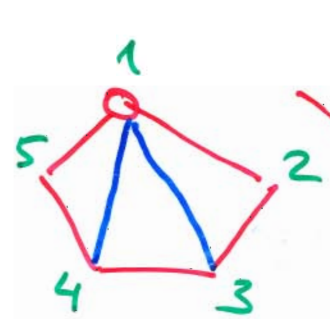
$$= \log t_{13} + \log t_{14} - \log t_{24} - \log t_{35} - \log t_{25}$$

$$z_1 = \frac{t_{13}\, t_{14}}{t_{24}\, t_{25}\, t_{35}}$$

$n=4$



Height function

$h_5 = \sum_{\circ}\Big($  $-$  $\Big)$

$$= \log(z_2 z_4)\log(z_3 z_5) - \log(z_1 z_3)\log(z_1 z_4)$$

$+$ perms